

Demazure character formula for semi-infinite flag manifolds

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Abstract

We provide a proof that every Schubert manifold of a semi-infinite flag manifold is projectively normal. This gives us an interpretation of a Demazure module of a global Weyl module of a current Lie algebra as the space of (the dual of) the global sections of a semi-infinite Schubert manifold. Moreover, we give geometric realizations of Feigin-Makedonskyi's generalized Weyl modules, and the $t = \infty$ specialization of non-symmetric Macdonald polynomials.

Introduction

Semi-infinite flag manifold is a variant of the affine flag manifold that encodes representation theory of affine Lie algebras [13]. It also admits an interpretation as the space of rational maps, and therefore plays a role in the computation of quantum K -theory of flag varieties. This latter direction was pursued by a series of papers by Braverman-Finkelberg [4, 6, 5], that leads to the proof of fundamental properties such as a proof of its normality, rationality of its singularities, an analogue of the Borel-Weil theorem, the computation of quantum J -functions (extending the work of Givental-Lee [18]), and its connection with q -Whittaker functions.

The aim of this paper is two-folds: one is to extend their cohomology formula of a line bundle to include some naturally twisted sheaves, and the other is to generalize their results to all Schubert manifolds so that the situation becomes more satisfactory from a representation-theoretic view-point. It turns out that such an extension provides a natural realization of certain specializations of non-symmetric Macdonald polynomials, together with difference equations characterizing them, generalizing their links to the representation theory of current algebras as discovered by Braverman-Finkelberg [6], Lenart-Naito-Sagaki-Schilling-Shimozono [27, 26, 28], Cherednik-Orr [11], Naito-Nomoto-Sagaki [29], and Feigin-Makedonskyi [14].

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To explain what we mean by this, we introduce more notation: Let G be a simply-connected simple algebraic group, let W be its Weyl group with the set $\{s_i\}_{i \in \mathbf{I}}$ of simple reflections, let Λ be its set of weights, and let Λ_+ be the set of dominant weights. Let Q^\vee be the coroot lattice of G . Then, we have the space of rational maps \mathcal{Q} from \mathbb{P}^1 to G/B , and its subspace formed as the closure of the set of rational maps whose value at 0 lands on a Schubert variety corresponding to $w \in W$. They carry a natural line bundle $\mathcal{O}(\lambda)$ corresponding to each $\lambda \in \Lambda$. Associated to G , we have a current algebra $\mathfrak{g}[z] := \text{Lie } G \otimes_{\mathbb{C}} \mathbb{C}[z]$ and its Iwahori subalgebra \mathfrak{I} . The Lie algebra $\mathfrak{g}[z]$ also possesses a natural representation $W(\lambda)$ for each $\lambda \in \Lambda_+$, that is called a global Weyl module (we set $W(\lambda) := \{0\}$ if $\lambda \in \Lambda \setminus \Lambda_+$). Kashiwara [23] defined its Demazure submodule $W(\lambda)_w$ to be the cyclic \mathfrak{I} -submodule generated by a vector with weight $w\lambda \in \Lambda$ for each $w \in W$. As they are graded, we have their character $\text{ch } W(\lambda)_w$, valued in $\mathbb{C}((q))[\Lambda]$.

Theorem A (\doteq Theorem 4.12 + Theorem 4.13). *For each $\lambda \in \Lambda$ and $w \in W$, we have:*

1. *The indscheme $\mathcal{Q}(w)$ is normal, and projectively normal;*
2. *We have the following isomorphism as \mathfrak{I} -modules:*

$$H^i(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))^* \cong \begin{cases} W(\lambda)_w & (i = 0, \lambda \in \Lambda_+) \\ \{0\} & (\text{otherwise}) \end{cases};$$

3. *For each $i \in \mathbf{I}$ so that $s_i w > w$, we have $\text{ch } W(\lambda)_{s_i w} = D_i(\text{ch } W(\lambda)_w)$, where D_i is a Demazure operator acting on $\mathbb{C}((q))[\Lambda]$;*
4. *We also have a Demazure operator D_β for each $\beta \in Q^\vee$ so that $\langle \beta, w\alpha \rangle \leq 0$ for every positive root α , that are mutually commutative. We have*

$$D_\beta(\text{ch } W(\lambda)_w) = q^{\langle \beta, w\lambda \rangle} \cdot \text{ch } W(\lambda)_w. \quad (0.1)$$

We remark that Theorem A 2)–4) can be regarded as a semi-infinite analogue of the Demazure character formula due to Demazure-Joseph-Kumar in the ordinary setting (see Kumar [25] VIII), that contains difference equations (0.1) characterizing them.

Theorem B (= Theorem 5.1 + Corollary 5.2). *For each $w \in W$ and $\lambda \in \Lambda_+$, the module $W(\lambda)_w$ admits a free action of a certain polynomial ring and its specialization to \mathbb{C} gives the Feigin-Makedonskyi module $W_{w\lambda}$. In particular, we have*

$$\Gamma(\text{Fl}_G^{\frac{\infty}{2}}(w), \mathcal{O}_{\text{Fl}_G^{\frac{\infty}{2}}(w)}(\lambda))^* \cong W_{w\lambda},$$

where $\text{Fl}_G^{\frac{\infty}{2}}(w)$ is a variant of $\mathcal{Q}(w)$.

Cherednik-Orr [11] obtained a recursive formula of non-symmetric Macdonald polynomials specialized to $t = \infty$. The comparison with our construction yields:

Theorem C (= Corollary 6.10). *For each $\lambda \in \Lambda_+$ and $w \in W$, there exists an $(\mathbf{I} \times \mathbb{G}_m)$ -equivariant sheaf $\mathcal{E}_w(\lambda)$ so that*

$$\mathrm{ch} H^0(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^* = \left(\prod_{i \in \mathbf{I}} \prod_{k=1}^{\langle \alpha_i^\vee, \lambda_w \rangle} \frac{1}{1 - q^k} \right) \cdot E_{-w\lambda}^\dagger(q^{-1}, \infty),$$

where λ_w is a dominant weight determined by λ and w , and $E_{-w\lambda}^\dagger(q, t)$ is the (bar-conjugate of the) non-symmetric Macdonald polynomial (see §5). In addition, we have $H^i(\mathcal{Q}(w), \mathcal{E}_w(\lambda)) = \{0\}$ for $i > 0$.

We remark that the vector space $H^0(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^*$ is a cyclic \mathfrak{I} -module (Lemma 6.7). One thing missing here at the moment is an analogue of Theorem B in the setting of Theorem C.

In the course of its proof, we present an analogue of the Kodaira vanishing theorem (Proposition 6.4) along the line of Kumar [25]. Some particular instances of our results are two formulas, one of which is [11] Proposition 2.5:

Corollary D (= Corollary 5.5). *For each $\lambda \in \Lambda_+$, we have the following relations between different specializations of non-symmetric Macdonald polynomials:*

$$\begin{aligned} D_{w_0}(E_{w_0\lambda}^\dagger(q^{-1}, \infty)) &= E_{w_0\lambda}^\dagger(q, 0) \\ D_{w_0t_\beta}(E_{w_0\lambda}^\dagger(q, 0)) &= q^{\langle \beta, \lambda \rangle} \cdot E_{w_0\lambda}^\dagger(q^{-1}, \infty), \end{aligned}$$

where $w_0 \in W$ is the longest element, $\beta \in Q^\vee$ satisfies $\langle \beta, \alpha_i \rangle < 0$ for each $i \in \mathbf{I}$, and t_β is the translation element in the affine Weyl group $W \ltimes Q^\vee$.

The organization of this note is as follows. The first two sections are preliminary materials on current algebra representations and semi-infinite flag manifolds, respectively. We provide proofs of some facts for which the author was unable to find appropriate references. The third section is a preparatory observation that the semi-infinite flag manifold must be actually projectively normal. The fourth section contains a proof of Theorem A through algebraic manipulations. Taking account into the works of Braverman-Finkelberg [4, 6, 5], the idea is supported by the fact that the Demazure character formula is in fact equivalent to the normality of Schubert varieties in the classical case. The fifth section contains a proof of Theorem B. Its main argument gives a simple (to the author's point of view) explanation of a result Feigin-Makedonskyi-Orr [15] (cf. Naito-Nomoto-Sagaki [29]). The sixth section is about Theorem C, that is a geometric interpretation of the intertwiners in the theory of non-symmetric Macdonald polynomials at $t = \infty$ due to Cherednik-Orr [11] (which can be also seen as a semi-infinite analogue of the $t = 0$ specialization of twisted non-symmetric Macdonald polynomials obtained by Sanderson and Ion [33, 19]).

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1 Preparatory materials

Throughout this note, a variety is a separated reduced scheme of finite type over \mathbb{C} , and its points are closed points unless otherwise stated.

A vector space is always a \mathbb{C} -vector space, and a graded vector space refers to a \mathbb{Z} -graded vector space whose grading is bounded from the below and each of its graded piece is finite-dimensional. For a graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ or its completion $M = \prod_{i \in \mathbb{Z}} M_i$, we define its dual as $M^* := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, where $\text{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$ is understood to have degree $-i$. (We sometimes deal with the graded completion of the dual of a graded module, that is not a graded module in our sense. In such an occasion, we regrade the module in an opposite way if necessary.) We define the graded dimension of a graded vector space as

$$\text{gdim } M := \sum_{i \in \mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}((q)).$$

For each n, k , we denote by $\mathbb{C}[\mathbb{A}^{(n)}]_{\leq k}$ the degree $\leq k$ -part of the symmetric polynomial ring of n -variables (of their degrees one).

1.1 Generality

Let G be a connected, simply connected simple algebraic group over \mathbb{C} , and let B and H be a Borel subgroup and a maximal torus of G so that $H \subset B$. We set $U (= [B, B])$ to be the unipotent radical of B and let U^- be the opposite unipotent subgroup of U with respect to H . We denote the Lie algebra of an algebraic group by German letters. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E , we denote its set of $\mathbb{C}[z]$ -valued points by $E[z]$, its set of $\mathbb{C}[[z]]$ -valued points by $E[[z]]$, and its set of $\mathbb{C}(z)$ -valued points by $E(z)$.

Let $\Lambda := \text{Hom}_{gr}(H, \mathbb{C}^\times)$ be the weight lattice of H , let $\Delta \subset \Lambda$ be the set of roots, let $\Delta_+ \subset \Delta$ be the set of roots belonging to \mathfrak{b} , and let $\Pi \subset \Delta_+$ be the set of simple roots. We set $\Delta_- := -\Delta_+$. For $\lambda, \mu \in \Lambda$, we define $\lambda \geq \mu$ if and only if $\lambda - \mu \in \mathbb{Z}_{\geq 0} \Delta_+$. Let Q^\vee be the dual lattice of Λ with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times \Lambda \rightarrow \mathbb{Z}$. We define $\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q_+^\vee \subset Q^\vee$ be the set of non-negative integer span of Π^\vee . We set $\Lambda_+ := \{\lambda \in \Lambda \mid \langle \alpha, \lambda \rangle \geq 0, \forall \alpha \in \Pi^\vee\}$. Let r be the rank of G and we set $\mathbf{I} := \{1, 2, \dots, r\}$. We fix bijections $\mathbf{I} \cong \Pi \cong \Pi^\vee$ so that $i \in \mathbf{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i \in W$ corresponding to α_i . We also have a reflection $s_\alpha \in W$ corresponding to $\alpha \in \Delta_+$. Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element. Let $\Delta_{\text{aff}} := \Delta \times \mathbb{Z}\delta \cup \{m\delta\}_{m \neq 0}$ be the untwisted affine root system of Δ with its positive part $\Delta_+ \subset \Delta_{\text{aff}, +}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\text{aff}} := \Pi \cup \{\alpha_0\}$, and $\mathbf{I}_{\text{aff}} := \mathbf{I} \cup \{0\}$, where ϑ is the highest root of Δ_+ . We set $W_{\text{aff}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathbf{I}_{\text{aff}}\}$, where s_0 is the reflection with respect to α_0 . Sending $s_0 \mapsto s_\vartheta$ (and $s_i \mapsto s_i$ for $i \in \mathbf{I}$) induces a group homomorphism $W_{\text{aff}} \ni w \mapsto \overline{w} \in W$. Together with the normalization $t_{-\vartheta^\vee} := s_\vartheta s_0$ (for the coroot ϑ^\vee of ϑ), we introduce the translation element $t_\beta \in W_{\text{aff}}$ for each $\beta \in Q^\vee$.

Let $\text{ev}_0 : G[z] \rightarrow G$ be the evaluation map at $z = 0$. For each $\mathbf{J} \subset \mathbf{I}$, we have a Coxeter subgroup $W_{\mathbf{J}} \subset W$ whose simple reflections are $\{s_i \mid i \in \mathbf{J}\}$ and a parabolic subgroup $B \subset P_{\mathbf{J}} \subset G$ whose Weyl group (of the Levi part)

is naturally identified with W_J . We have a unique connected closed subgroup $G[z] \not\supset \mathbf{I}_0 \subset G(z)$ that contains $\mathbf{I} (= \mathbf{I}_\emptyset)$. For each $i \in \mathbf{I}$, we denote by B_i^0 the intersection of \mathbf{I} with the semi-simple Levi component L_i^0 of \mathbf{I}_i that is stable by the adjoint H -action.

For each $\lambda \in \Lambda_+$, we denote by $V(\lambda)$ (or $V_G(\lambda)$ in case we specify G) the irreducible finite-dimensional \mathfrak{g} -module with its highest weight λ . It is standard that we have a unique non-zero vector $v_{w\lambda} \in V(\lambda)$ of weight $w\lambda$ up to scalar for each $w \in W$.

Let $\varpi_1, \dots, \varpi_r \in \Lambda_+$ be the dual basis of Π^\vee . For $\lambda \in \Lambda_+$, we expand it as

$$\lambda = \sum_{i=1}^r \lambda_i \varpi_i \quad \text{with} \quad \lambda_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 1 \leq i \leq r$$

and define $|\lambda| := \sum_{i=1}^r \lambda_i$ and $\lambda! := \prod_{i=1}^r \lambda_i!$. We also identify λ with a composition $(\lambda_1, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r$. Using this identification, we define

$$\mathbb{C}[\mathbb{A}^{(\lambda)}] := \bigotimes_{i=1}^r \mathbb{C}[x_{i,1}, \dots, x_{i,\lambda_i}]^{\mathfrak{S}_{\lambda_i}} \subset \bigotimes_{i=1}^r \mathbb{C}[x_{i,1}, \dots, x_{i,\lambda_i}] =: \mathbb{C}[\mathbb{A}^\lambda].$$

Let $\widehat{\mathfrak{g}}$ be the untwisted affine Kac-Moody Lie algebra arising from \mathfrak{g} , and let $\mathfrak{g}[z] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]$ be the current algebra of \mathfrak{g} . We have natural inclusions $\mathfrak{g} \subset \mathfrak{g}[z] \subset \widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d \subset \widehat{\mathfrak{g}}$ be the Cartan subalgebra that prolongs $\mathfrak{h} \subset \mathfrak{g}$ with a convention that $[K, \widehat{\mathfrak{g}}] = 0$ and d is the degree operator of $\mathfrak{g}[z]$. We equip a \mathbb{Z} -grading of $\mathfrak{g}[z]$ by setting $\deg \xi \otimes z^m = m$ for every $\xi \in \mathfrak{g} \setminus \{0\}$ (this is the grading induced by the d -action). We note that $U(\mathfrak{g}[z])$ is not a graded vector space in our sense.

Let $\mathbb{K} := \mathbb{C}(t)$ and let U_t be the quantum loop algebra of $\widehat{\mathfrak{g}}$ (see e.g. [23] 2.1). It has the positive part $U_t^+ \subset U_t$, the Cartan part $U_t^0 \subset U_t$, and the classical part $U_t^\flat \subset U_t$. We have their $\mathbb{C}[t]$ -integral lattices $\mathbf{U}_t^\flat, \mathbf{U}_t^+, \mathbf{U}_t^0$ so that

$$\mathbf{U}_t^+ \otimes_{\mathbb{C}[t]} \mathbb{C}_0 \cong U([\mathfrak{J}, \mathfrak{J}]), \quad \mathbf{U}_t^0 \otimes_{\mathbb{C}[t]} \mathbb{C}_0 \subset U(\mathfrak{h} \oplus \mathbb{C})^\wedge, \quad \text{and} \quad \mathbf{U}_t^\flat \otimes_{\mathbb{C}[t]} \mathbb{C}_0 \subset U(\mathfrak{g})^\wedge,$$

where $U(\mathfrak{h} \oplus \mathbb{C})^\wedge$ and $U(\mathfrak{g})^\wedge$ are the integral weight idempotents completions of $U(\mathfrak{h} \oplus \mathbb{C})$ and $U(\mathfrak{g})$ and their inclusions are dense, respectively. We set $U_t^{\geq 0} := U_t^+ U_t^0 \subset U_t$. The algebra U_t also admits an $\exp d$ -action (by embedding it into a quantum algebra of Kac-Moody type) that commute with U_t^\flat , so that the degree $\exp(m)$ -part of U_t corresponds to the degree m -part of $U(\mathfrak{g}[z, z^{-1}])$ for each $m \in \mathbb{Z}$. We regrade this degree $\exp(m)$ -part of U_t^+ as the degree m -part.

For each $0 \neq \lambda \in \Lambda_+$ and $x \in \mathbb{C}$, we sometimes regard $V(\lambda)$ as an irreducible $\mathfrak{g}[z]$ -module via the Lie algebra quotient map $\mathfrak{g}[z] \rightarrow \mathfrak{g}[z]/(z-x)\mathfrak{g}[z] \cong \mathfrak{g}$, that we denote by $V(\lambda, x)$. (We note that $V(0, x) = V(0, 0)$ for every $x \in \mathbb{C}$.) For a graded \mathfrak{J} -module M , we define its character as

$$\text{ch } M := \sum_{\lambda \in \Lambda} e^\lambda \text{gdim Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, M) \in \mathbb{Q}((q))[\Lambda].$$

We replace \mathfrak{h} with $\mathbb{K}[Q^\vee] \subset U_t$ to define a character of a $U_t^{\geq 0}$ -module (with the multiplicative action on \mathbb{C}_λ). For two such modules M and N , we denote $\text{ch } M \leq \text{ch } N$ if the corresponding inequality holds for every coefficient of $q^k e^\lambda$ ($k \in \mathbb{Z}, \lambda \in \Lambda$). Each $V(\lambda)$ ($\lambda \in \Lambda_+$) admits a lift $V_t(\lambda)$ into a U_t^\flat -module so

that $\text{ch } V(\lambda, 0) = \text{ch } V_t(\lambda)$ by further extending to a $U_t^{\geq 0}$ -module concentrated in degree 0.

Let $X := G/B$ be the flag variety of G , that we sometimes denote by X_G . For each $\lambda \in \Lambda$, we have a line bundle $G \times^B \lambda$, that we denote by $\mathcal{O}_X(\lambda)$. For each $w \in W$, we have a B -orbit $\mathbb{O}(w) \subset X$ obtained as $B\dot{w}B/B \subset X$ with a unique T -fixed point x_w , where $\dot{w} \in N_G(H)$ is a lift of w (so that $\mathbb{O}(w)$ is independent of the choice). We set $X(w) := \overline{\mathbb{O}(w)}$. It is well-known that $\dim X(w) = \ell(w)$. For $w, w' \in W$, we denote by $w > w'$ if and only if $X(w) \supset X(w')$.

1.2 Current algebras

Definition 1.1 (integrable module). A $\mathfrak{g}[z]$ -module M is said to be integrable if and only if M decomposes into a direct sum of finite-dimensional \mathfrak{g} -modules. Let $\mathfrak{g}[z]\text{-mod}$ be the category of finitely generated integrable $\mathfrak{g}[z]$ -module. For each $\lambda \in \Lambda_+$, let $\mathfrak{g}[z]\text{-mod}^{\leq \lambda}$ be the fullsubcategory of $\mathfrak{g}[z]\text{-mod}$ whose object is isomorphic to a direct sum of \mathfrak{g} -modules in $\{V(\mu)\}_{\mu \leq \lambda}$.

Definition 1.2 (projective modules and global Weyl module). For each $\lambda \in \Lambda_+$, we define the non-restricted projective module $P(\lambda)$ as

$$P(\lambda) := U(\mathfrak{g}[z]) \otimes_{U(\mathfrak{g})} V(\lambda).$$

Let $P(\lambda; \mu)$ be the largest $\mathfrak{g}[z]$ -module quotient of $P(\lambda)$ so that

$$\text{Hom}_{\mathfrak{g}}(V(\gamma), P(\lambda; \mu)) = \{0\} \quad \text{if} \quad \gamma \not\leq \mu. \quad (1.1)$$

We define the global Weyl module $W(\lambda)$ of \mathfrak{g} to be $P(\lambda; \lambda)$.

Lemma 1.3. *The projective module $P(\lambda)$, its quotient $P(\lambda; \mu)$ and global Weyl modules $W(\lambda)$ can be regarded as graded modules with a simple head $V(\lambda, 0)$ sitting at degree 0 (for $\lambda, \mu \in \Lambda_+$).*

Proof. Straight-forward from the construction. \square

Theorem 1.4 (Chari-Loktev [8], Fourier-Littelmann [17], Naoi [32]). *For each $\lambda \in \Lambda_+$, it holds:*

1. *the module $P(\lambda)$ is the projective cover of $V(\lambda, x)$ as an integrable $\mathfrak{g}[z]$ -module for every $x \in \mathbb{C}$;*
2. *the module $W(\lambda)$ admits a free action of $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ induced by the $U(\mathfrak{h}[z])$ -action on the \mathfrak{h} -weight λ -part of $W(\lambda)$, that commutes with the $\mathfrak{g}[z]$ -action;*
3. *the natural grading structure of $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ respects the grading of $W(\lambda)$;*

For each $x \in \mathbb{A}^{(\lambda)}$, we have a specialization $W(\lambda, x) := W(\lambda) \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_x$.

4. *$W(\lambda, x) \cong W(\lambda, y)$ as \mathfrak{g} -modules for each $x, y \in \mathbb{A}^{(\lambda)}$;*
5. *if $x \in \mathbb{A}^{(\lambda)}$ is the orbit of $|\lambda|$ -distinct points, then we have*

$$W(\lambda, x) \cong \bigotimes_{i=1}^r \bigotimes_{j=1}^{\lambda_i} W(\varpi_i, x_{i,j}).$$

Here $(x_{i,1}, \dots, x_{i,\lambda_i}) \in \mathbb{A}^{\lambda_i}$ corresponds to x (up to \mathfrak{S}_{λ_i} -action).

Proof. The assertion 1) follows by the definition through the Frobenius reciprocity. As explained in Chari-Ion [7, 2.8–2.10], the simply-laced cases of the assertions 2)–5) are contained in [17] and the non simply-laced cases are contained in [32]. \square

Definition 1.5 (local Weyl module). For each $\lambda \in \Lambda_+$ and $x \in \mathbb{A}^{(\lambda)}$, we call $W(\lambda, x)$ (in Theorem 1.4) the local Weyl module supported on x .

Theorem 1.6 (Chari-Loktev, Fourier-Littelmann, Naoi). *For each $\lambda \in \Lambda_+$, there exists a $U_t^{\geq 0}$ -module $W_t(\lambda)$ with a $\mathbb{C}[t]$ -lattice so that its reduction by $t = 1$ yields $W(\lambda)$. In particular, we have $\text{ch } W_t(\lambda) = \text{ch } W(\lambda)$.*

Proof. The first assertion ultimately relies on Kashiwara [21, 23] as the existence of the global basis on $W_t(\lambda)$ (cf. [22]). The second assertion after $\otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0$ follows by the isomorphism of local Weyl modules through Theorem 1.4 4) (originally conjectured by Chari-Pressley [9] §4). As we have a natural morphism from $W(\lambda)$ to the reduction $W'(\lambda)$ of $W_t(\lambda)$ modulo $(t-1)$, the comparison of the $U(\mathfrak{h}[z])$ -actions by Theorem 1.4 2) and Beck-Nakajima [1] Theorem 4.16 implies the second and the third assertions. The third assertion is also derived from the comparison of Naito-Sagaki [31] Theorem 6.4.1 and Chari-Ion [7] Proposition 4.3. \square

2 Semi-infinite Schubert manifolds

We review the quasi-map realization of semi-infinite flag manifold of G , for which the basic references are Finkelberg-Mirković [16] and Feigin-Finkelberg-Kuznetsov-Mirković [12].

We have W -equivariant isomorphisms $H^2(X, \mathbb{Z}) \cong \Lambda$ and $H_2(X, \mathbb{Z}) \cong Q^\vee$. This identifies the ample cone of X with $\Lambda_+ \subset \Lambda$ and the effective cone of X with Q_+^\vee . A quasi-map (f, D) is a map $f : \mathbb{P}^1 \rightarrow X$ together with a Π -colored effective divisor

$$D = \sum_{\alpha \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee) \alpha^\vee \otimes (x) \in Q^\vee \otimes_{\mathbb{Z}} \text{Div } \mathbb{P}^1 \quad \text{with} \quad m_x(\alpha^\vee) \in \mathbb{Z}_{\geq 0}.$$

For $i \in \mathbf{I}$, we set $D_i := \langle D, \varpi_i \rangle \in \text{Div } \mathbb{P}^1$. We call D the defect of the quasi-map (f, D) . Here we define the degree of the defect by

$$|D| := \sum_{\alpha \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee) \alpha^\vee \in Q_+^\vee.$$

Theorem 2.1 (Drinfeld-Plücker data over fields, see Braverman-Gaitsgory [3] 1.1.2). *Let \mathbb{K} be an overfield of \mathbb{C} . Then, the set of collections $\{\mathbb{K}v_\lambda\}_{\lambda \in \Lambda_+}$ of lines in $V(\lambda) \otimes_{\mathbb{C}} \mathbb{K}$ so that*

$$v_\lambda \otimes_{\mathbb{K}} v_\mu \in \mathbb{K}v_{\lambda+\mu} \subset V(\lambda+\mu) \otimes_{\mathbb{C}} \mathbb{K} \subset V(\lambda) \otimes_{\mathbb{C}} V(\mu) \otimes_{\mathbb{C}} \mathbb{K} \quad \text{for each } \lambda, \mu \in \Lambda_+$$

is in bijection with the set of closed \mathbb{K} -points of X . \square

Definition 2.2 (Drinfeld-Plücker data). Consider a collection $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in \Lambda_+}$ of inclusions $\psi_\lambda : \mathcal{L}^\lambda \hookrightarrow V(\lambda) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ of line bundles \mathcal{L}^λ over \mathbb{P}^1 . The data \mathcal{L} is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of G -modules

$$\eta_{\lambda, \mu} : V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu)$$

induces an isomorphism

$$\eta_{\lambda,\mu} \otimes \text{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \xrightarrow{\cong} \psi_{\lambda}(\mathcal{L}^{\lambda}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \psi_{\mu}(\mathcal{L}^{\mu})$$

for every $\lambda, \mu \in \Lambda$.

For each $\beta \in Q_+^{\vee}$, we set

$$\mathcal{Q}(X, \beta) := \{f : \mathbb{P}^1 \rightarrow X \mid \text{quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta\},$$

where $f_*[\mathbb{P}^1]$ is the class of the image of \mathbb{P}^1 multiplied by the degree of $\mathbb{P}^1 \rightarrow \text{Im } f$. We sometimes denote $\mathcal{Q}(X, \beta)$ by $\mathcal{Q}(\beta)$ in case there is no danger of confusion, and also for various varieties and indschemes of the form $\mathcal{Q}_?(X, w, ?)$ defined in the below. The topology of this space arises from:

Theorem 2.3 (Drinfeld, see Finkelberg-Mirković [16]). *The variety $\mathcal{Q}(X, \beta)$ is isomorphic to the variety formed by isomorphism classes of the DP-data $\mathcal{L} = \{(\psi_{\lambda}, \mathcal{L}^{\lambda})\}_{\lambda \in \Lambda_+}$ such that $\deg \mathcal{L}^{\lambda} = -\langle \beta, \lambda \rangle$.*

For each $\beta, \beta' \in Q_+^{\vee}$, we have an embedding

$$\iota^{\beta, \beta'} : \mathcal{Q}(\beta) \hookrightarrow \mathcal{Q}(\beta + \beta'),$$

that simply adds the defect by $\beta' \otimes (\infty)$. We set $\mathcal{Q}(X) := \varinjlim_{\beta} \mathcal{Q}(X, \beta)$ and call it the (indscheme model of the) semi-infinite flag manifold of G . We have a natural $G[z]$ -action on \mathcal{Q} that preserves the defect.

Let $\mathcal{Q}_0(X)$ denote the subspace of $\mathcal{Q}(X)$ whose defect is supported outside of $0 \in \mathbb{P}^1$. We have a natural evaluation map

$$\text{ev}_0 : \mathcal{Q}_0 \longrightarrow X,$$

that is $G[z]$ -equivariant. It restricts to $\mathcal{Q}_0(\beta) \subset \mathcal{Q}(\beta)$ for each $\beta \in Q_+^{\vee}$. For each $w \in W$, we define $\mathcal{Q}(X, w) := \overline{\text{ev}_0^{-1}(X(w))}$ and call it the semi-infinite Schubert manifold.

For each $\lambda \in \Lambda$, we have a $G[z]$ -equivariant line bundle $\mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)$ (and its pro-object $\mathcal{O}_{\mathcal{Q}}(\lambda)$) obtained by the (tensor product of the) pull-backs $\mathcal{O}_{\mathcal{Q}(\beta)}(\varpi_i)$ of the i -th $\mathcal{O}(1)$ via the embedding

$$\mathcal{Q}(\beta) \hookrightarrow \prod_{i \in \mathbf{I}} \mathbb{P}(V(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[z]_{\leq \langle \beta, \varpi_i \rangle}), \quad (2.1)$$

for each $\beta \in Q_+^{\vee}$ (see e.g. [6] §2.1).

We set $\mathcal{O}_{\mathcal{Q}(w)}(\lambda)$ and $\mathcal{O}_{\mathcal{Q}(w, \beta)}(\lambda)$ ($\beta \in Q_+^{\vee}$) to be the pullback of $\mathcal{O}_{\mathcal{Q}}(\lambda)$ to $\mathcal{Q}(w)$ and $\mathcal{Q}(w, \beta)$, respectively. For each $\beta \in Q_+^{\vee}$, let us consider an affine closed subset $\mathbf{I}^{\leq \beta} \subset \mathbf{I}$ so that its action on $V(\lambda) \otimes \mathbb{C}[z]$ contains matrix entries of degree at most $\langle \beta, \lambda \rangle$. We have $\mathbf{I}^{\leq \beta} \cdot \mathbf{I}^{\leq \beta'} \subset \mathbf{I}^{\leq \beta + \beta'}$ for each $\beta, \beta' \in Q_+^{\vee}$ and $\mathbf{I} = \bigcup_{\beta \in Q_+^{\vee}} \mathbf{I}^{\leq \beta}$. Taking account into (2.1), we deduce an ind-action

$$\mathbf{I}^{\leq \gamma} \cdot \mathcal{Q}(\beta) \longrightarrow \mathcal{Q}(\beta + \gamma) \quad \text{for each } \beta, \gamma \in Q_+^{\vee}$$

that is compatible with $\iota^{\beta, \beta'}$.

The ind-action of \mathbf{I} on \mathcal{Q} preserves $\mathcal{Q}(w)$ for each $w \in W$ since $\text{ev}_0(\mathbf{I}) = B$.

Theorem 2.4 (Braverman-Finkelberg [4] Theorem 1.2). *For each $\beta \in Q_+^\vee$, the variety $\mathcal{Q}(\beta)$ is normal.* \square

By taking the formal expansion of the map along 0, we have a natural $G[z]$ -equivariant embedding $\mathcal{Q} \hookrightarrow \mathbf{Q}$ into an infinite type scheme \mathbf{Q} that contains $G[[z]]/(H \cdot U[[z]])$ as its open subset. The scheme \mathbf{Q} admits a natural $G[[z]]$ -action extending that of $G[z]$ (that is realized by replacing $\mathbb{C}[z]_{\leq k}$ with $\mathbb{C}[[z]]$ in (2.1)). We have a $G[[z]]$ -subscheme $\mathbf{Q}_0 \subset \mathbf{Q}$ that has an evaluation at $z = 0$. Hence, we have $\mathbf{Q}(w)$ in a parallel fashion to \mathcal{Q} . They admit a natural action of the completed version \mathbf{I}^\wedge of \mathbf{I} (we also define the completed version \mathbf{I}_i^\wedge of \mathbf{I}_i for each $i \in \mathbf{I}_{\text{aff}}$). By construction, we have $\mathcal{Q}(w) = \mathcal{Q} \cap \mathbf{Q}(w)$ for each $w \in W$.

Lemma 2.5. *The ind-action of \mathbf{I} on $\mathcal{Q}(w)$ has a Zariski open dense orbit.*

Proof. The inclusion $\mathcal{Q}(w) \subset \mathbf{Q}(w)$ is dense, and the latter has an open dense orbit with respect to the \mathbf{I}^\wedge -action. Therefore, there exists an \mathbf{I} -indorbit whose closure in $\mathbf{Q}(w)$ contains an open dense \mathbf{I}^\wedge -orbit. Such an \mathbf{I} -indorbit must be open dense as required. \square

We define

$$H^i(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda)) := \varprojlim H^i(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)) \quad \text{for every } i \in \mathbb{Z}.$$

Theorem 2.6 (Braverman-Finkelberg [6, 5]). *For each $\lambda \in \Lambda$, we have a natural isomorphism*

$$H^i(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda))^* \cong \begin{cases} W(\lambda) & (i = 0, \lambda \in \Lambda_+) \\ \{0\} & (\text{otherwise}) \end{cases}$$

as a graded $\mathfrak{g}[z]$ -module (where the grading arises from the loop rotation). \square

Corollary 2.7. *The line bundle $\mathcal{O}_{\mathcal{Q}}(\lambda)$ is very ample if and only if $\langle \alpha_i^\vee, \lambda \rangle > 0$ for every $i \in \mathbf{I}$.*

Proof. Thanks to (2.1), we know that

$$\mathcal{Q} \hookrightarrow \prod_{i \in \mathbf{I}} \mathbb{P}\Gamma(V(\varpi_i) \otimes \mathbb{C}[z]).$$

Theorem 1.4 1) asserts that $\Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\varpi_i))^* \cong W(\varpi_i) \rightarrow V(\varpi_i) \otimes \mathbb{C}[z]$ is a surjective $\mathfrak{g}[z]$ -module homomorphism. Therefore, we have

$$\mathcal{Q} \hookrightarrow \prod_{i \in \mathbf{I}} \mathbb{P}\Gamma(V(\varpi_i) \otimes \mathbb{C}[z]) \dashleftarrow \mathbb{P}\Gamma(\otimes_{i \in \mathbf{I}} W(\varpi_i)),$$

that prolongs to a commutative diagram of the embeddings of \mathcal{Q} . By Theorem 1.4 5), we have

$$(\otimes_{i \in \mathbf{I}} W(\varpi_i)) \otimes_{\mathbb{C}[z_i; i \in \mathbf{I}]} \mathbb{C}(z_i; i \in \mathbf{I}) \cong W(\rho) \otimes_{\mathbb{C}[z_i; i \in \mathbf{I}]} \mathbb{C}(z_i; i \in \mathbf{I}),$$

and \mathcal{Q} contains a $H \cdot U[z]$ -fixed vector v corresponding to a constant loop so that $\text{Stab}_{G[z]}v = H \cdot U[z]$, that commutes with adding arbitrary defects. Together with the G -equivariance, this implies $\mathcal{Q} \subset \mathbb{P}(W(\rho))$. In particular, $\mathcal{O}_{\mathcal{Q}}(\rho) = \bigotimes_{i \in \mathbf{I}} \mathcal{O}_{\mathcal{Q}}(\varpi_i)$ is a very ample sheaf of \mathcal{Q} . In general, $\mathcal{O}_{\mathcal{Q}}(\lambda - \rho)$ has a non-zero global section by Theorem 2.6, and we have an embedding $\mathcal{O}_{\mathcal{Q}}(\rho) \hookrightarrow \mathcal{O}_{\mathcal{Q}}(\lambda - \rho) \otimes \mathcal{O}_{\mathcal{Q}}(\rho) \cong \mathcal{O}_{\mathcal{Q}}(\lambda)$ of (pro-)line bundles on \mathcal{Q} , that yields the if statement.

Only if statement is clear since the restriction of $\mathcal{O}_{\mathcal{Q}}(\lambda)$ to the subspace of constant loops is $\mathcal{O}_X(\lambda)$, that is base point free if and only if $\langle \alpha_i^\vee, \lambda \rangle > 0$ for every $i \in \mathbf{I}$. \square

3 Ind-scheme structures on $\mathcal{Q}(w)$

We retain the setting of the previous section.

Definition 3.1 (Ind-systems). Let $w \in W$. An increasing sequence of closed subsets

$$\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \mathfrak{X}_3 \subset \cdots \subset \mathcal{Q}(w)$$

of finite type is said to be an ind-system of $\mathcal{Q}(w)$ if $\bigcup_{k \geq 1} \mathfrak{X}_k = \mathcal{Q}(w)$ and for every $N \in \mathbb{Z}$, there exists $\beta \in Q_+^\vee$ so that $\mathfrak{X}_N \subset \mathcal{Q}(w, \beta)$, and for every $\beta \in Q_+^\vee$, there exists $N \in \mathbb{Z}$ so that $\mathcal{Q}(w, \beta) \subset \mathfrak{X}_N$.

Lemma 3.2. Let $w \in W$ and $\lambda \in \Lambda$. Fix an ind-system $\{\mathfrak{X}_k\}_{k \geq 1}$ of $\mathcal{Q}(w)$. For each $i \in \mathbb{Z}$, we have

$$\varprojlim_k H^i(\mathfrak{X}_k, \mathcal{O}_{\mathfrak{X}_k}(\lambda)) = H^i(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda)).$$

Proof. The LHS is the limit through a projective system $H^i(\mathfrak{X}_{k+1}, \mathcal{O}_{\mathfrak{X}_{k+1}}(\lambda)) \rightarrow H^i(\mathfrak{X}_k, \mathcal{O}_{\mathfrak{X}_k}(\lambda))$ for each $k \geq 1$. By the condition of an ind-system, we find $\beta_1, \beta_2 \in Q_+^\vee$ for each $M \gg N \in \mathbb{Z}_{\geq 0}$ so that

$$\begin{aligned} H^i(\mathcal{Q}(w, \beta_2), \mathcal{O}_{\mathcal{Q}(w, \beta_2)}(\lambda)) &\rightarrow H^i(\mathfrak{X}_M, \mathcal{O}_{\mathfrak{X}_M}(\lambda)) \\ &\rightarrow H^i(\mathcal{Q}(w, \beta_1), \mathcal{O}_{\mathcal{Q}(w, \beta_1)}(\lambda)) \rightarrow H^i(\mathfrak{X}_N, \mathcal{O}_{\mathfrak{X}_N}(\lambda)). \end{aligned}$$

We also find $M, N \in \mathbb{Z}_{\geq 0}$ with the same maps if we fix $\beta_2 \gg \beta_1$. Therefore, two pro-systems factor through each other, which implies

$$\varprojlim_k H^i(\mathfrak{X}_k, \mathcal{O}_{\mathfrak{X}_k}(\lambda)) = \varprojlim_\beta H^i(\mathcal{Q}(w, \beta), \mathcal{O}_{\mathcal{Q}(w, \beta)}(\lambda))$$

as required. \square

Theorem 3.3. The ind-scheme $\mathcal{Q}(w_0) = \mathcal{Q}$ is projectively normal.

Proof. The homogeneous coordinate ring $R(w_0)$ of $\mathcal{Q}(w_0)$ is obtained as the graded completion of its \mathbb{G}_m -finite part $R^\#(w_0) = \bigoplus_{\lambda \in \Lambda_+} W(\lambda)^*$ (cf. [6] Theorem 1.5).

Let us fix a collection of non-zero elements $y = \{y_i\}_{i \in \mathbf{I}}$ so that $y_i \in W(\varpi_i)^*$ for each $i \in \mathbf{I}$. Consider the ring R_y obtained from $R(w_0)$ through the localization of y . As we fix y , there exists $\beta_0 \in Q_+^\vee$ so that the image of y_i in $H^0(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\varpi_i))$ is non-zero for each $i \in \mathbf{I}$ when $\beta > \beta_0$ (we remind that each $\mathcal{Q}(\beta)$ is integral as being normal). Then, the image of y defines an affine ring $R(\beta)_y$ obtained by the localization of the homogeneous coordinate ring of $\mathcal{Q}(\beta)$. By the definition of the homogeneous coordinate ring, we can form the ring $R(\beta)_y$ only using $H^0(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda))$ for $\lambda \gg 0$. By the Serre's vanishing theorem, such a rearrangement guarantees the projective system to be surjective, and consequently $R(\beta)_y$ is a quotient of R_y . In such a circumstance, R_y is integral as each $R_y(\beta)$ is so. Now we assume to the contrary to deduce contradiction, so that we assume R_y is not normal. We have a monic equation $P(X)$ with coefficients in R_y that has a solution in $\text{Frac } R_y$, but not in R_y . A solution of $P(X) = 0$ is written as $X = \frac{a}{b}$ by $a, b \in R_y$. For $\beta \gg 0$, all the coefficients of the equation $P(X)$, and $a, b \in R_y$ go to non-zero elements of $R(\beta)_y$. By Theorem 2.4, we find that $a/b = c(\beta) \in R(\beta)_y$ for $\beta \gg 0$. Taking the inverse

limit yields an element in R_y that maps to $\{c(\beta)\}_{\beta \gg 0}$. Therefore, we conclude that R_y is normal. By the definition of DP-data and the embedding (2.1) (cf. the proof of Corollary 2.7), the open sets $\cap_{i \in \mathbf{I}} \{y_i \neq 0\}$ cover the whole \mathcal{Q} , and hence \mathcal{Q} is normal.

It remains to show that the dual of the multiplication map $W(\lambda + \mu) \longrightarrow W(\lambda) \otimes W(\mu)$ is injective for each $\lambda, \mu \in \Lambda_+$ (here we used the fact that the normality of \mathcal{Q} is equivalent to that of $\mathbb{P}_{\mathcal{Q}}(\bigoplus_{i \in \mathbf{I}} \mathcal{O}_{\mathcal{Q}}(\varpi_i)^\vee)$). Here this map extends the (dual) multiplication map $V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu)$, that is uniquely determined up to scalar as \mathfrak{g} -modules. Note also that $\mathbb{C}[\mathbb{A}^{(\lambda)}] \otimes \mathbb{C}[\mathbb{A}^{(\mu)}]$ is a free $\mathbb{C}[\mathbb{A}^{(\lambda+\mu)}]$ -module of rank $\frac{(\lambda+\mu)!}{\lambda! \mu!}$. Thanks to Theorem 1.4 5), a generic specialization along $x \in \mathbb{A}^{(\lambda+\mu)}$ yields an inclusion

$$\begin{array}{ccc} W(\lambda + \mu) \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda+\mu)}]} \mathbb{C}_x & \longrightarrow & (W(\lambda) \otimes W(\mu)) \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda+\mu)}]} \mathbb{C}_x \\ \cong \downarrow & & \downarrow \cong \\ \bigotimes_{i \in \mathbf{I}} \bigotimes_{j=1}^{\langle \alpha_i^\vee, \lambda+\mu \rangle} W(\varpi_i, x_{i,j}) & \hookrightarrow & \left(\bigotimes_{i \in \mathbf{I}} \bigotimes_{j=1}^{\langle \alpha_i^\vee, \lambda+\mu \rangle} W(\varpi_i, x_{i,j}) \right)^{\oplus \frac{(\lambda+\mu)!}{\lambda! \mu!}} \end{array}$$

where $\{x_{i,j}\}$ is a set of points in \mathbb{C} determined by the configuration of x (as the map is non-zero and a non-zero $\mathfrak{g}[z]$ -module endomorphism of $W(\varpi_i, x)$ must be an isomorphism). Since any $\mathbb{C}[\mathbb{A}^{(\lambda+\mu)}]$ -submodule of a free $\mathbb{C}[\mathbb{A}^{(\lambda+\mu)}]$ -module of finite rank has no torsion element (that is supported on some closed subset of $\mathbb{A}^{(\lambda+\mu)}$), we conclude the inclusion $W(\lambda + \mu) \longrightarrow W(\lambda) \otimes W(\mu)$ as required. \square

Definition 3.4 (Demazure modules). For $\lambda \in \Lambda_+$ and $w \in W$, we have a unique vector $v_{w\lambda} \in V(\lambda) \subset W(\lambda)$ of \mathfrak{h} -weight $w\lambda$ up to scalar. We define

$$W(\lambda)_w := U(\mathfrak{J})v_{w\lambda} \subset W(\lambda)$$

and call it the Demazure submodule of $W(\lambda)$. By Theorem 1.6, we also define a $U_t^{\geq 0}$ -submodule $W_t(\lambda)$ generated by a vector with its U_t^0 -weight $w\lambda$ at degree 0. We note that $W(\lambda) = W(\lambda)_{w_0}$ and $W_t(\lambda) = W_t(\lambda)_{w_0}$.

Corollary 3.5 (of the proof of Proposition 3.3). *For each $\lambda, \mu \in \Lambda_+$ and $w \in W$, we have an injective map $m_{\lambda, \mu}^w : W(\lambda + \mu)_w \longrightarrow W(\lambda)_w \otimes W(\mu)_w$.* \square

For each $w \in W$, we define a ring (that generalizes $R(w_0)$ in the proof of Proposition 3.3)

$$R^\#(w) := \bigoplus_{\lambda \in \Lambda_+} W(\lambda)_w^*,$$

where the product structure is given by Corollary 3.5. Let $R(w)$ denote the \mathbb{G}_m -graded completion of $R(w)$, taken Λ_+ -degree-wise.

Corollary 3.6. *The ring $R(w_0)$ is normal.* \square

Corollary 3.7 (of the proof of Proposition 3.3). *The ind-scheme $\mathcal{Q}(w)$ is projectively normal if the ring $R(w)$ is normal and $R^\#(w)$ defines a dense subring of the projective coordinate ring of $\mathcal{Q}(w)$.*

Proof. Our ind-system is equivalent to these obtained by cutting out by the degrees by its definition (cf. (2.1)). Therefore, if $R^\#(w)$ is a dense subring of the projective coordinate ring of $\mathcal{Q}(w)$, then the latter is $R(w)$. \square

4 Main Results

We continue to work in the setting of the previous section.

Definition 4.1 (Demazure operator). For each $i \in \mathbf{I}_{\text{aff}}$, we define a linear operator on $\mathbb{C}((q))[\Lambda]$ by

$$D_i(q^m e^\lambda) := q^m \frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}} \quad \text{for each } m \in \mathbb{Z} \text{ and } \lambda \in \Lambda,$$

where we formally put $q = e^\delta$. For $w \in W_{\text{aff}}$, we fix a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of w and set

$$D_w := D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_\ell}.$$

Theorem 4.2 (Demazure-Joseph, cf. Kumar [25] §VIII). *We have:*

1. For each $w \in W_{\text{aff}}$, the Demazure operator D_w is independent of the reduced expression;
2. For each $\lambda \in \Lambda$ and $w \in W$, we have

$$\sum_{i \geq 0} (-1)^i \text{ch } H^i(X(w), \mathcal{O}_{X(w)}(\lambda))^* = D_w(e^\lambda);$$

3. For each $\lambda \in \Lambda_+$ and $w \in W$, we have $H^0(X(w), \mathcal{O}_{X(w)}(\lambda))^* \cong U(\mathfrak{b})v_{w\lambda}$ as B -modules and $H^i(X(w), \mathcal{O}_{X(w)}(\lambda)) = \{0\}$ for $i > 0$;
4. For each $w \in W$, the restriction through $X(w) \subset X$ induces a B -module inclusion $H^0(X(w), \mathcal{O}_{X(w)}(\lambda))^* \subset V(\lambda)$. \square

Lemma 4.3. For each $\lambda \in \Lambda_+$ and $w \in W$, the space $\Gamma(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))^*$ contains a non-zero vector of weight $w\lambda$ arising from $\Gamma(X(w), \mathcal{O}_{X(w)}(\lambda))^*$.

Proof. We have $0 \neq v_{w\lambda} \in \Gamma(X(w), \mathcal{O}_{X(w)}(\lambda))^* \subset \Gamma(X, \mathcal{O}_X(\lambda))^*$ by Theorem 4.2 3) and 4). We have an inclusion $X(w) \subset \mathcal{Q}(w)$ of constant quasimaps with their defects supported on ∞ , that presents a section of ev_0 . The degree 0-part of the map $\mathcal{Q} \rightarrow \mathbb{P}\Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda))^*$ represents the image of the evaluation map $\mathcal{Q}_0 \rightarrow X$. In particular, we have $[v_{w\lambda}] \in X(w) \subset \mathcal{Q}(w)$. Being a unique vector of weight $w\lambda$ at degree 0 in $W(\lambda)$, the dual vector v^* of $v_{w\lambda}$ in $\Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda))$ is uniquely determined up to a scalar. Since v^* defines a non-zero regular function on $\mathcal{Q}(w)$, it survives through the restriction to $\Gamma(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))$. Hence, we deduce $v_{w\lambda} \in (\text{Im } \Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda)) \rightarrow \Gamma(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda)))^*$. Therefore, $v_{w\lambda}$ must prolong to $\Gamma(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))^*$ as required. \square

Lemma 4.4. Let V be a graded \mathfrak{g} -module with finitely many distinct \mathfrak{h} -weights. Let $E \subset V$ be its \mathfrak{b} -submodule. For each $i \in \mathbf{I}$, we have

$$U(\mathfrak{sl}_2)E \subset H^0(\mathbb{P}^1, SL(2) \times^B E)^* \quad \text{and} \quad \text{ch } U(\mathfrak{p}_i)E \leq D_i(\text{ch } E),$$

where the latter equality holds if and only if V has a finite \mathfrak{p}_i -filtration so that the associated graded of its induced \mathfrak{b} -filtration on E is a direct sum of irreducible \mathfrak{sl}_2 -modules (corresponding to $i \in \mathbf{I}$) and one-dimensional representations of \mathfrak{b} of weight γ so that $\langle \alpha_i^\vee, \gamma \rangle > 0$. The analogous assertion also holds for a $U_t^\mathfrak{b}$ -module V and its $(U_t^\mathfrak{b} \cap U_t^{\geq 0})$ -submodule E .

Proof. Since E is assumed to be \mathfrak{b} -stable, we have $U(\mathfrak{p}_i)E = U(\mathfrak{sl}_2)E$, where \mathfrak{sl}_2 is the \mathfrak{h} -stable semi-simple Levi component of \mathfrak{p}_i . Hence, we replace \mathfrak{p}_i and P_i with \mathfrak{sl}_2 and $SL(2)$ during this proof.

We have a natural inclusion $U(\mathfrak{sl}_2)E \subset H^0(\mathbb{P}^1, SL(2) \times^B E)^*$ coming from the restriction to the \mathfrak{sl}_2 -highest weight part of E regarded as a fiber at B/B . The inequality is easy to verify when V is irreducible, and we deduce the inequality by the Euler-Poincaré principle in general (as the both sides are additive with respect to a short exact sequence).

In case E admits such a filtration, each graded piece define subquotients of $SL(2) \times^B E \subset SL(2) \times^B V$ of the form $F \otimes \mathcal{O}_{\mathbb{P}^1} = F \otimes \mathcal{O}_{\mathbb{P}^1}(\lambda)$ (for a \mathfrak{sl}_2 -module F) or $V(\lambda) \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(\lambda)$ with $\lambda \in \mathbb{Z}_{\geq 0}\varpi = \Lambda_+$. In the all cases, we have $H^1(\mathbb{P}^1, \bullet) = \{0\}$, and a successive applications of short exact sequences yields if part of the middle assertion.

We prove the only if part of the middle assertion. For each $k \geq 0$, we define $V[k]$ to be the \mathfrak{sl}_2 -direct summand of V whose highest weight is $k\varpi$ (via the restriction). Consider the filtration

$$\{0\} \subsetneq V(N) \subsetneq V(N-1) \subsetneq V(N-2) \subsetneq \cdots \subsetneq V(0) = V,$$

where $V(k) = V(k+1) \oplus V[k]$ for each $k \geq 0$ (and we have $V(k) = \{0\}$ for $k \gg 0$). Note that each $V[k]$ and $V(k)$ inherits the natural grading and the \mathfrak{h} -module structure. We define $E(k) := E \cap V(k)$ for each $k \geq 0$. Each $E(k)$ is stable by the \mathfrak{b}_i^0 -action. We assume N' to be the largest number so that $E(N')/E(N'+1)$ is not a direct sum of \mathfrak{sl}_2 -modules and one-dimensional \mathfrak{b}_i^0 -modules of weight $\mathbb{Z}_{>0}\varpi$ to deduce contradiction. We have

$$\text{ch } H^0(\mathbb{P}^1, SL(2) \times^B E(N'))^* - \text{ch } U(\mathfrak{sl}_2)E(N') > \text{ch } H^1(\mathbb{P}^1, SL(2) \times^B E(N'))^*$$

from the \mathfrak{b}_i^0 -invariance of the $E(N')$ and the hypothesis (with the help of Euler-Poincaré principle). This is the same as an inequality

$$\text{ch } U(\mathfrak{sl}_2)E(N') < D_i(\text{ch } E(N')). \quad (4.1)$$

For each $v \in E \setminus E(N')$ so that $v' \in U(\mathfrak{b}_i^0)v$ has \mathfrak{sl}_2 -weight $k\varpi$ for $k \geq N'$, we have $v' \in E(N')$ by a weight counting. In particular, we have

$$(v + U(\mathfrak{sl}_2)E(N')) \cap \bigoplus_{k < N'} V[k] \neq \emptyset.$$

This forces

$$(U(\mathfrak{sl}_2)E) / (U(\mathfrak{sl}_2)E(N')) \cong U(\mathfrak{sl}_2)(E/E(N')) \subset \bigoplus_{k < N'} V[k].$$

In particular, we have an inequality

$$\text{ch } (U(\mathfrak{sl}_2)E) / (U(\mathfrak{sl}_2)E(N')) = \text{ch } U(\mathfrak{sl}_2)(E/E(N')) \leq D_i(\text{ch } E/E(N')) \quad (4.2)$$

The inequalities (4.1) and (4.2) results in

$$\text{ch } (U(\mathfrak{sl}_2)E) < D_i(\text{ch } E(N')) + D_i(\text{ch } E/E(N')) = D_i(\text{ch } E).$$

Therefore, we have no possible choice of N' . Hence the only if part of the middle assertion follows.

Since the integrable representation theory of $U_t(\mathfrak{sl}_2)$ (with t being generic) and $U(\mathfrak{sl}_2)$ are the same, exactly the same proof works in the quantum setting as required. \square

Definition 4.5. For $w \in W$ and $i \in \mathbf{I}_{\text{aff}}$, we define $\overline{s_i w} >_q w$ if we have $s_i w > w$ (when $i \in \mathbf{I}$) or $w^{-1}\vartheta \notin \Delta^+$ (when $i = 0$).

Theorem 4.6 (LNSSS-I [27] §6). *For every $w, v \in W$, there exists a sequence $i_1, i_2, \dots, i_\ell \in \mathbf{I}_{\text{aff}}$ so that*

$$w = \overline{s_{i_1} s_{i_2} \cdots s_{i_\ell} v} >_q \overline{s_{i_2} \cdots s_{i_\ell} v} >_q \cdots >_q \overline{s_{i_\ell} v} >_q v. \quad (4.3)$$

Proof. The relation $>_q$ without taking the projection $W_{\text{aff}} \rightarrow W$ generates an order in W_{aff} . It is a variant of the quantum (or generic) Bruhat order in the sense that the weak Bruhat order is different from the Bruhat order (cf. Ishii-Naito-Sagaki [20] Appendix A.3 and Bjorner-Brenti [2]). Therefore, the assertion is included in Lenart-Naito-Sagaki-Schilling-Shimozono [27] §6. \square

Theorem 4.7 (Kashiwara [23] 2.8, Naito-Sagaki [31] §5). *Let $\lambda \in \Lambda_+$ and let $w \in W$. For each $i \in \mathbf{I}_{\text{aff}}$ such that $\overline{s_i w} >_q w$, we have an identity*

$$D_i(\text{ch } W_t(\lambda)_w) = q^{\delta_{i,0} \langle \vartheta^\vee, w\lambda \rangle} \cdot \text{ch } W_t(\lambda)_{\overline{s_i w}}.$$

Proof. In view of Lemma 4.4, the assertion follows if the \mathfrak{sl}_2 -crystal (corresponding to $i \in \mathbf{I}$) structure of $W_t(\lambda)_w$ inside $W_t(\lambda)_{s_i w}$ is a disjoint union of genuine \mathfrak{sl}_2 -crystals and Demazure crystals (it is a crystal with one element with weight γ so that $\langle \alpha_i^\vee, \gamma \rangle > 0$ in this case).

The assertion on crystal itself follows by [23] Lemma 2.7 as the crystal basis there is equal to these of $W_t(\lambda)_{s_i w}$ as \mathfrak{sl}_2 -crystals (cf. [23] §2.5, see also [31] proof of Proposition 5.1.1). \square

Corollary 4.8. *Let $\lambda \in \Lambda_+$ and let $w \in W$. For each $i \in \mathbf{I}_{\text{aff}}$ such that $\overline{s_i w} >_q w$, we have an identity*

$$D_i(\text{ch } W(\lambda)_w) = q^{\delta_{i,0} \langle \vartheta^\vee, w\lambda \rangle} \cdot \text{ch } W(\lambda)_{\overline{s_i w}}.$$

Proof. By using a $\mathbb{C}[t]$ -lattice of $W_t(\lambda)_w \subset W_t(\lambda)$, the specialization map $t \rightarrow 1$ yields an \mathfrak{J} -module inclusion $W'(\lambda)_w \subset W(\lambda)$. Since $W'(\lambda)_w$ shares a vector $v_w \lambda$ with $W(\lambda)_w$, we have $W(\lambda)_w \subset W'(\lambda)_w$. In particular, we have $\text{ch } W(\lambda)_w \leq \text{ch } W'(\lambda)_w$ for each $w \in W$. By Theorem 1.6, this is an equality for $w = w_0$.

We prove the assertion on induction on w from w_0 . Let $i \in \mathbf{I}_{\text{aff}}$ so that $\overline{s_i w} >_q w$. Since we have $W(\lambda)_w = W'(\lambda)_w$, we have

$$U(\mathfrak{J}_i)W(\lambda)_w = U(\mathfrak{J}_i)W'(\lambda)_w \subset W'(\lambda)_{\overline{s_i w}}. \quad (4.4)$$

Here $W'(\lambda)_{\overline{s_i w}}$ is the specialization of a module $W_t(\lambda)_{\overline{s_i w}}$ by setting $t = 1$ in their $\mathbb{C}[t]$ -lattice spanned by the global bases. By the proof of Theorem 4.7, every global basis element of $W_t(\lambda)_{\overline{s_i w}}$ is labeled by a highest weight element viewed as a \mathfrak{sl}_2 -crystal belongs to $W_t(\lambda)_w$. In view of Lemma 4.4 and Theorem 4.7, a $U_t(\mathfrak{sl}_2)$ -highest weight vector of $W_t(\lambda)_{\overline{s_i w}}$ is contained in $W_t(\lambda)_w$ with grading shift $\langle \vartheta^\vee, w\lambda \rangle$ when $i = 0$. By the comparison of characters, we deduce that the dimension of the space of $U_t(\mathfrak{sl}_2)$ -highest weight vectors of $W_t(\lambda)_{\overline{s_i w}}$ with given weight and degree coincides with the number of highest weight elements of the Demazure crystal of $W_t(\lambda)_{\overline{s_i w}}$ with the same weight and degree (that is finite). By the multiplication rule of the global bases (see e.g. [23] Definition 2.4 iii)), we deduce that a sum of global basis elements (of a fixed weight) corresponding to

non-highest weight elements viewed as \mathfrak{sl}_2 -crystal never gives rise to a non-zero $U(\mathfrak{sl}_2)$ -highest weight vector by reduction mod $(t-1)$. Therefore, we cannot have a \mathfrak{sl}_2 -highest weight vector in $W'(\lambda)_{\overline{s_i w}} \setminus W'(\lambda)_w$ with a given \mathfrak{h} -weight and degree. It follows that $U(\mathfrak{sl}_2)W'(\lambda)_w = W'(\lambda)_{\overline{s_i w}}$. Thus, the inclusion in (4.4) is in fact an equality.

By the PBW theorem, we have $W(\lambda)_{\overline{s_i w}} \cong U(\mathfrak{sl}_2)W(\lambda)_w$. Now Theorem 4.7 implies

$$D_i(\text{ch } W(\lambda)_w) = D_i(\text{ch } W'(\lambda)_w) = q^{\delta_{i,0} \langle \vartheta^\vee, w\lambda \rangle} \cdot \text{ch } W'(\lambda)_{\overline{s_i w}} = q^{\delta_{i,0} \langle \vartheta^\vee, w\lambda \rangle} \cdot \text{ch } W(\lambda)_{\overline{s_i w}},$$

which proceeds the induction as required. \square

Proposition 4.9. *For each $w \in W$, the ring $R(w)$ is normal.*

Proof. For each $\beta \in Q_+^\vee$ and $\lambda \in \Lambda_+$, we have a $\mathfrak{g}[z]$ -module embedding $W(\lambda) \hookrightarrow W(\lambda)$ induced by the multiplication of $z^{\langle \beta, \varpi_i \rangle}$ for $W(\varpi_i) \rightarrow V(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[z]$ for each $i \in \mathbf{I}$ through Corollary 3.5 (it is a product of $\langle \beta, \varpi_i \rangle$ -th power of a degree λ_i primitive generator of $\mathbb{C}[\mathbb{A}^{(\lambda_i)}] \subset \mathbb{C}[\mathbb{A}^{(\lambda)}]$ in Theorem 1.4 2)). This endomorphism is the same (up to scalar) as the action of a lift of $t_\beta \in W_{\text{aff}}$ to $H(z)$ in view of the embedding (2.1) (with an extension of the scalar to $\mathbb{C}(z)$ if necessary). In addition, it also corresponds to the twist of cyclic vectors of Demazure modules corresponding to D_{t_β} in accordance with Corollary 4.8. Therefore, it extends to an inclusion $W(\lambda)_w \hookrightarrow W(\lambda)_w$ for each $w \in W$. It further gives rise to a surjection $R(w) \twoheadrightarrow R(w)$ of algebra induced by each $\beta \in Q_+^\vee$. Hence, the definition of $R(w)$ can be naturally extended to $w \in W_{\text{aff}}$, with the difference by a translation part gives rise to an isomorphic algebra with degree twists in accordance with Corollary 4.8. (These are rephrasements of the inclusions $\mathcal{Q} \hookrightarrow \mathcal{Q}$ and $\mathcal{Q}(w) \hookrightarrow \mathcal{Q}(w)$ given by twisting defects supported on 0, though the latter is yet to be established.) In view of this, we can prove the assertion by induction on $>_q$ using Theorem 4.6. The case $w = w_0$ is Corollary 3.6. We assume the assertion for $w \in W$ and find $i \in \mathbf{I}_{\text{aff}}$ so that $\overline{s_i w} >_q w$.

The algebra $R(w)$ admits a B_i^0 -module structure. In addition, we can write $R(w) := \varprojlim_m R(w)_m$, where $R(w)_m$ is a suitable $(H \cdot B_i^0)$ -stable graded quotient of $R(w)_m$ that is a $(\Lambda_+$ -graded componentwise) finite dimensional vector space of bounded degrees (thanks to the degree-wise Mittag-Leffler condition, only the topology is a matter of concern). We form an ind-vector bundle $\mathcal{R}_i(w) := \varprojlim_m SL(2) \times^{B_i^0} R(w)_m^*$ over \mathbb{P}^1 . Fix $x \in \mathbb{P}^1(\mathbb{C})$, and find a local coordinate t_x of x . We have $\mathbb{C}[t_x]_{(0)} \cong \mathcal{O}_{\mathbb{P}^1, x}$ as a ring, where (0) denote the localization along $t_x = 0$. The stalk of $\mathcal{R}_i(w)$ at x is isomorphic to the scalar extension $R(w) \otimes_{\mathbb{C}} \mathbb{C}[t_x]_{(0)}$, and hence is normal. Now we have

$$H^0(\mathbb{P}^1, \mathcal{R}_i(w)) = \bigcap_{x \in \mathbb{P}^1(\mathbb{C})} R(w) \otimes_{\mathbb{C}} \mathbb{C}[t_x]_{(0)} \subset \text{Frac}(R(w) \otimes_{\mathbb{C}} \mathbb{C}(\mathbb{P}^1)).$$

Since the intersection of normal rings that shares the same fraction field is normal (by the definition of integral closure), we conclude that the ring $H^0(\mathbb{P}^1, \mathcal{R}_i(w))$ is normal. By construction, we have $W(\lambda)_{\overline{s_i w}} = U(\mathfrak{p}_i)W(\lambda)_w \subset W(\lambda)$ for each $\lambda \in \Lambda_+$ (with a possible degree twist of $W(\lambda)$). By Lemma 4.4, we deduce

$$R_{\clubsuit}^\#(\overline{s_i w}) = U(\mathfrak{p}_i)R^\#(w) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{R}_i(w)), \quad (4.5)$$

where $R_{\clubsuit}^{\#}(\overline{s_i w})$ is obtained by a degree twist of $W(\lambda)_{\overline{s_i w}}$ by $\langle \vartheta^{\vee}, w\lambda \rangle$ when $i = 0$. In particular, we have an inclusion $R^{\#}(\overline{s_i w}) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{R}_i(w))$ of algebras. Therefore, the comparison of Corollary 4.8 with (4.5) forces $R(\overline{s_i w}) \cong H^0(\mathbb{P}^1, \mathcal{R}_i(w))$ (through Lemma 4.4). This shows that $R(\overline{s_i w})$ is a normal ring, and the induction proceeds. \square

Lemma 4.10. *Let $\beta \in Q_+^{\vee}$, $w \in W$, and $i \in \mathbf{I}$ so that $s_i w > w$. We have a map $q_i : P_i \times^B \mathcal{Q}(w, \beta) \rightarrow \mathcal{Q}(s_i w, \beta)$. Similarly, we have a map $P_i \times^B \mathcal{Q}(w) \rightarrow \mathcal{Q}(s_i w)$ that we denote by the same letter.*

Proof. The variety $\mathcal{Q}(\beta)$ is irreducible, and so is its open subset $\mathcal{Q}_0(\beta)$. Since $X(w)$ is connected, we deduce that $\mathcal{Q}_0(w, \beta)$, and hence $\mathcal{Q}(w, \beta)$ is irreducible. As $\mathcal{Q}(\beta)$ is projective, so is $\mathcal{Q}(w, \beta)$. Therefore, the image of q_i is irreducible and projective. In addition, we have $\mathcal{Q}_0(s_i w, \beta) \subset \mathcal{Q}_0(\beta) \cap \text{Im } q_i$, that is actually an open dense subset of $\text{Im } q_i$. Therefore, we conclude $\mathcal{Q}(s_i w, \beta) = \text{Im } q_i$, that implies the first assertion. The second assertion is now clear. \square

Lemma 4.11. *Let $\beta \in Q_+^{\vee}$ and $w \in W$ so that $w^{-1}\vartheta \notin \Delta^+$. We have a map $q_0 : \text{SL}(2) \times^{B_0^0} \overline{B_0^0 \mathcal{Q}(w, \beta)} \rightarrow \mathcal{Q}(s_{\vartheta} w, \beta + \gamma)$ for some $\gamma \leq 2\vartheta^{\vee}$ that is independent of β . Similarly, we have a map $\mathbf{I}_0 \times^{\mathbf{I}} \mathcal{Q}(w) \rightarrow \mathcal{Q}$ (that we denote by the same letter) whose image is $\mathcal{Q}(s_{\vartheta} w)$ with an appropriate twist by the defect at 0.*

Proof. We have a map

$$\text{SL}(2) \times \prod_{i \in \mathbf{I}} \mathbb{P}(V(\varpi_i) \otimes \mathbb{C}[z]_{\leq m}) \longrightarrow \prod_{i \in \mathbf{I}} \mathbb{P}(V(\varpi_i) \otimes z^{-m_i} \mathbb{C}[z]_{\leq m+2m_i}),$$

where $m_i := \langle \vartheta^{\vee}, \varpi_i \rangle$ for $i \in \mathbf{I}$. This map does not preserve $\mathcal{Q}(\beta)$ (in usual and ind-senses), but we see that the point $[v_{w\varpi_i}]$ is sent to $[v_{s_{\vartheta} w\varpi_i} \otimes z^{\langle \vartheta^{\vee}, w\varpi_i \rangle}]$. Since the B_0^0 -action on $[v_{s_{\vartheta} w\varpi_i}]$ is open dense in $\text{SL}(2)[v_{s_{\vartheta} w\varpi_i}] \cong \mathbb{P}^1$, Lemma 2.5 implies that the image of $\mathcal{Q}(w, \beta)$ by the multiplication by $\text{SL}(2)$ is contained in $\mathcal{Q}(s_{\vartheta} w)$ if we twist the degrees by $\langle w^{-1}\vartheta^{\vee}, \varpi_i \rangle$ for the i -th component of the embedding. Since ϑ^{\vee} is the highest short coroot, we have $\vartheta^{\vee} \geq w^{-1}\vartheta^{\vee} \geq -\vartheta^{\vee}$, regardless the value of β . Now, adjusting the defect (at 0) to the image of $[v_{s_{\vartheta} w\varpi_i} \otimes z^{\langle \vartheta^{\vee}, w\varpi_i \rangle}]$ and taking the limit $\beta \rightarrow \infty$ yields $\mathcal{Q}(s_{\vartheta} w)$ as the image of q_0 . This proves the both assertions as required. \square

Theorem 4.12. *For each $\lambda \in \Lambda$ and $w \in W$, it holds:*

1. *we have the following isomorphisms as \mathfrak{I} -modules:*

$$H^i(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))^* \cong \begin{cases} W(\lambda)_w & (i = 0, \lambda \in \Lambda_+) \\ \{0\} & (\text{otherwise}) \end{cases};$$

2. *the restriction map $\Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\lambda)) \longrightarrow \Gamma(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))$ is surjective;*

3. *the indscheme $\mathcal{Q}(w)$ is normal and projectively normal.*

Proof. We first consider the case $w = w_0$. Then, the first assertion follows by Theorem 2.6. The second assertion is trivial, and the third assertion follows by Proposition 3.3.

Since shifting by adding defects at $0 \in \mathbb{P}^1$ gives an isomorphic pair of ind-schemes, we prove the assertion by induction on $>_q$ using Theorem 4.6. We assume that the assertions hold for $w \in W$ and fix $i \in \mathbf{I}_{\text{aff}}$ so that $\overline{s_i w} >_q w$. For the sake of simplicity, we denote $\overline{s_i w}$ by $s_i w$ during this proof.

We set $\mathcal{Q}^+(w, \beta) := \overline{B_i^0 \mathcal{Q}(w, \beta)}$ for each $\beta \in Q_+^\vee$. We have $\mathcal{Q}^+(w, \beta) = \mathcal{Q}(w, \beta)$ whenever $i \in \mathbf{I}$, and $\mathcal{Q}^+(w, \beta)$ forms an ind-structure of $\mathcal{Q}(w)$ by Lemma 4.11. Let us denote the image of q_0 in Lemma 4.11 by $\mathcal{Q}^+(s_\vartheta w, \beta)$ when $i = 0$. It defines an ind-structure of $\mathcal{Q}(s_\vartheta w)$ since it contains $\mathcal{Q}(s_\vartheta w, \beta - 2\vartheta^\vee)$ for $\beta \gg 0$ (by examining the proof of Lemma 4.11).

We have a \mathfrak{I} -module map

$$\eta : H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\lambda))^* \rightarrow H^0(\mathcal{Q}(w_0), \mathcal{O}_{\mathcal{Q}(w_0)}(\lambda))^* = W(\lambda)$$

arising from the dual of the restriction map. By Lemma 4.3, we have $W(\lambda)_{s_i w} \subset \text{Im } \eta$. In particular, we have

$$\text{ch } W(\lambda)_{s_i w} \leq \text{ch } H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\lambda))^* \quad (4.6)$$

By Corollary 4.8, the first assertion is equivalent to an isomorphism

$$H^k(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\lambda))^* \cong H^k(\mathbb{P}^1, SL(2) \times^{B_i^0} W(\lambda)_w) \quad \text{for each } k \in \mathbb{Z}. \quad (4.7)$$

By assumption and Lemma 3.2, we deduce that

$$H^k(\mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda)) \cong \varprojlim_{\beta} H^k(\mathcal{Q}^+(w, \beta), \mathcal{O}_{\mathcal{Q}^+(w, \beta)}(\lambda)) \quad \text{for each } k \in \mathbb{Z}. \quad (4.8)$$

We set $\mathcal{Q}^+(i, w, \beta) := SL(2, \mathbb{C}) \times^{B_i^0} \mathcal{Q}^+(w, \beta)$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{Q}^+(i, w, \beta) & \xrightarrow{q_i} & \mathcal{Q}^+(s_i w, \beta) \\ h_i \downarrow & & \downarrow \bar{h}_i \\ \mathbb{P}^1 & \xrightarrow{\bar{q}_i} & \text{pt} \end{array}$$

Claim A. *We have a (convergent) spectral sequence*

$$\varprojlim \mathbb{R}^u(\bar{h}_i)_*(\mathbb{R}^t(q_i)_* \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)}(\lambda)) \Rightarrow \varprojlim H^{u+t}(\mathcal{Q}^+(i, w, \beta), \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)}(\lambda)).$$

Proof. By (4.8) and the induction hypothesis, the pro-sheaf $\varprojlim \mathbb{R}^u(h_i)_* \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)}(\lambda)$ satisfies the Mittag-Leffler condition for each fixed degree. In addition, the effect of $(\bar{q}_i)_*$ changes the degree at most by $2 \langle \vartheta^\vee, \lambda \rangle$. Therefore, \varprojlim commutes with $\mathbb{R}^t(\bar{q}_i)_*$, and we deduce the Leray spectral sequence

$$\mathbb{R}^t(\bar{q}_i)_* \left(\varprojlim \mathbb{R}^u(h_i)_* \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)}(\lambda) \right) \Rightarrow \varprojlim H^{u+t}(\mathcal{Q}^+(i, w, \beta), \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)}(\lambda)).$$

Now q_i is a morphism with two sections $\mathcal{Q}^+(w, \beta) \hookrightarrow \mathcal{Q}^+(i, w, \beta)$ corresponding to $1, s_i \in SL(2)$. In particular, we can reinterpret $\mathbb{R}^t(\bar{q}_i)_*$ with the length two Čech complex representing the affine charts along sections. As each of such a piece represents $\otimes_{\mathbb{C}} \mathbb{C}[X]$ or $\otimes_{\mathbb{C}} \mathbb{C}[X, X^{-1}]$ that is independent of the pro-system, the (degree-wise) Mittag-Leffler condition along the fiber allows us to change the order of the spectral sequence. It converges as $E_2^{u, t} \neq \{0\}$ happens only if $0 \leq t \leq 1$. \square

We return to the proof of Theorem 4.12. By Claim A, we deduce a spectral sequence

$$H^t(\mathcal{Q}(s_i w), \mathbb{R}^u(q_i)_* \mathcal{O}_{\mathcal{Q}^+(i, w)}^+(\lambda)) \Rightarrow H^{u+t}(\mathbb{P}^1, SL(2) \times^{B_i^0} W(\lambda)_w). \quad (4.9)$$

Since the fiber of q_i is contained in \mathbb{P}^1 , it follows that $\mathbb{R}^k(q_i)_* \mathcal{O}_{\mathcal{Q}^+(w, \beta)} = \{0\}$ for $k \geq 2$. In addition, $\mathcal{Q}^+(i, w, \beta)$ is contained in $SL(2) \times^{B_i^0} \mathcal{Q}^+(s_i w, \beta)$, where the natural prolongization of q_i becomes a \mathbb{P}^1 -fibration. Therefore, the short exact sequence

$$0 \rightarrow \ker \rightarrow \mathcal{O}_{SL(2) \times^{B_i^0} \mathcal{Q}^+(s_i w, \beta)} \rightarrow \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)} \rightarrow 0$$

yields a part of the long exact sequence

$$0 = \mathbb{R}^1(q_i)_* \mathcal{O}_{SL(2) \times^{B_i^0} \mathcal{Q}^+(s_i w, \beta)} \rightarrow \mathbb{R}^1(q_i)_* \mathcal{O}_{\mathcal{Q}^+(i, w, \beta)} \rightarrow \mathbb{R}^2(q_i)_* \ker = 0,$$

where the last equality follows by the relative dimension counting. Therefore, we conclude that

$$H^k(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}^+(\lambda)) \cong H^k(\mathbb{P}^1, SL(2) \times^{B_i^0} W(\lambda)_w) \quad \text{for each } k \in \mathbb{Z}. \quad (4.10)$$

where $\mathcal{O}_{\mathcal{Q}(s_i w, \beta)}^+(\lambda) := (q_i)_* \mathcal{O}_{\mathcal{Q}^+(w, \beta)}(\lambda)$. By construction, we have an embedding $\mathcal{O}_{\mathcal{Q}(s_i w, \beta)}(\lambda) \hookrightarrow \mathcal{O}_{\mathcal{Q}(s_i w, \beta)}^+(\lambda)$ (and we can take their inverse limits by construction). In particular, taking their global sections yield:

$$\text{ch } H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\lambda))^* \leq \text{ch } H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}^+(\lambda))^*. \quad (4.11)$$

From (4.6), (4.11), and (4.10), we deduce that

$$\begin{aligned} \text{ch } W(\lambda)_{s_i w} &\leq \text{ch } H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\lambda))^* \leq \text{ch } H^0(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}^+(\lambda))^* \\ &= \text{ch } H^0(\mathbb{P}^1, SL(2) \times^{B_i^0} W(\lambda)_w)^*. \end{aligned} \quad (4.12)$$

Thanks to Corollary 4.8 (and Theorem 4.2), we derive that all the inequalities in (4.12) must be in fact an equality. This particularly shows that all the sections of $\mathcal{O}_{\mathcal{Q}(s_i w, \beta)}^+(\lambda)$ and $\mathcal{O}_{\mathcal{Q}(s_i w, \beta)}(\lambda)$ are the same by taking the inverse limit. A vector

$$f \in \text{Im} \left(\bigotimes_{i \in \mathbb{I}} W(\varpi_i)^* \rightarrow W(\rho)_{s_i w}^* \right) \subset \text{Im} (R(w_0) \rightarrow R(s_i w))$$

defines a section of $\Gamma(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(\rho))$. Hence, it defines an inclusion $\mathcal{O}_{\mathcal{Q}(s_i w)} \hookrightarrow \mathcal{O}_{\mathcal{Q}(s_i w)}(\rho)$ whose n -times repeated application gives $\mathcal{O}_{\mathcal{Q}(s_i w)} \hookrightarrow \mathcal{O}_{\mathcal{Q}(s_i w)}(n\rho)$. This leads to a map $\mathcal{O}_{\mathcal{Q}(s_i w)}^+ \hookrightarrow \mathcal{O}_{\mathcal{Q}(s_i w)}^+(n\rho)$. If f is homogeneous of degree $\geq -m$, then it defines an affine open subspace of each of $\mathcal{Q}(s_i w, \beta)$ for every $\beta \in \mathcal{Q}_+^\vee$ so that

$$\langle \beta, \varpi_i \rangle \geq m \quad \text{for each } i \in \mathbb{I} \quad (4.13)$$

by (2.1). Therefore, taking limit $n \rightarrow \infty$ is a localization to an affine open subset on $\mathcal{Q}(s_i w, \beta)$ whenever β satisfies (4.13). It induces an ind-affine subset $\mathcal{U}(f) = \varinjlim_{\beta} \mathcal{U}(f, \beta)$. As the localization is flat, it commutes with \varprojlim and Γ

as the condition (4.13) is clearly satisfied for every $\beta' > \beta$ whenever β satisfies (4.13). Therefore, we conclude that

$$\begin{aligned}\Gamma(\mathfrak{U}(f), \mathcal{O}_{\mathcal{Q}(s_i w)}(n\rho)) &= \varinjlim_{n \rightarrow \infty} \Gamma(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}(n\rho)) \\ &= \varinjlim_{n \rightarrow \infty} \Gamma(\mathcal{Q}(s_i w), \mathcal{O}_{\mathcal{Q}(s_i w)}^+(n\rho)) = \Gamma(\mathfrak{U}(f), \mathcal{O}_{\mathcal{Q}(s_i w)}^+(n\rho)).\end{aligned}$$

Since every further localization to a point of $\mathfrak{U}(f)$ is realized as a projective system of local rings, we conclude that $\mathcal{O}_{\mathcal{Q}(s_i w)} \cong \mathcal{O}_{\mathcal{Q}(s_i w)}^+$ on $\mathfrak{U}(f)$ (as pro-sheaves) again by the flatness of the localization. Here we have $\bigcap_f \mathfrak{U}(f) = \emptyset$ by (2.1) as every point of $\mathcal{Q}(s_i w)$ is a point of $\mathcal{Q}(s_i w, \beta)$ for some $\beta \in Q_+^\vee$. This shows that $\mathcal{O}_{\mathcal{Q}(s_i w)} \cong \mathcal{O}_{\mathcal{Q}(s_i w)}^+$ as pro-sheaves. Therefore, we conclude (4.7) (or the first assertion). The second assertion follows as η must be an inclusion.

By the first assertion, $R^\#(w)$ is a dense subring of the projective coordinate ring of $\mathcal{Q}(w)$, and its graded completion is normal by Proposition 4.9. Therefore, Corollary 3.7 implies the projective normality of $\mathcal{Q}(w)$.

This proceeds the induction and completes the proof of Theorem 4.12. \square

Theorem 4.13 (Demazure character formula for $\mathcal{Q}(w)$). *For $\lambda \in \Lambda_+$, $\beta \in Q_+^\vee$, and $w, v \in W$ so that $\ell(wv) = \ell(w) + \ell(v)$ and $v^{-1}w^{-1}\beta \in Q_+^\vee$, we have*

$$D_{t_\beta w}(\text{ch } \Gamma(\mathcal{Q}(v), \mathcal{O}_{\mathcal{Q}(v)}(\lambda))^*) = q^{\langle \beta, wv\lambda \rangle} \cdot \text{ch } \Gamma(\mathcal{Q}(wv), \mathcal{O}_{\mathcal{Q}(wv)}(\lambda))^*.$$

In particular, we have

$$D_w(\text{ch } W(\lambda)_v) = \text{ch } W(\lambda)_{wv} \quad \text{if } w, v \in W.$$

Proof. By the definition of D_w , it suffice to prove $\text{ch } W(\lambda)_{s_\vartheta w} = q^{-\langle \vartheta^\vee, w\lambda \rangle} \cdot D_0(\text{ch } W(\lambda)_w)$ whenever $w^{-1}\vartheta \notin \Delta^+$, and $\text{ch } W(\lambda)_{s_i w} = D_i(\text{ch } W(\lambda)_w)$ whenever $s_i w > w$ for $i \in \mathbf{I}$. We have

$$D_i(\text{ch } W(\lambda)_w) = \text{ch } H^0(\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(w), \mathcal{O}_{\mathcal{Q}(w)}(\lambda))^*$$

by Corollary 4.8 and Theorem 4.12 (1). Therefore, if we take account into the fact that the lowest degree term $v_{s_i w \lambda}$ has degree count $\langle \vartheta^\vee, w\lambda \rangle$ when $i = 0$, then the result follows by induction. \square

5 Feigin-Makedonskyi modules

For each $\alpha \in \Delta$, we fix non-zero root vectors $e_\alpha \in \mathfrak{u}$ and $f_\alpha \in \mathfrak{u}^-$ of weight α and $-\alpha$, respectively. The following result is due to Feigin-Makedonskyi-Orr [15] (see also Naito-Nomoto-Sagaki [29] for its q -analogue), but we decided to include a proof as the author likes the proof in the below.

Theorem 5.1. *Let $\lambda \in \Lambda_+$ and $w \in W$. The module $W(\lambda)_w$ is free over $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ of rank $\dim W(\lambda, 0)$. In addition, the module $W(\lambda)_w \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0$ is generated by $v_{w\lambda}$ subject to the conditions:*

- $(h \otimes z)v_{w\lambda} = 0$ for every $h \in \mathfrak{h}$;

- In case $\alpha \in \Delta^+ \cap w\Delta^+$, we have

$$e_\alpha v_{w\lambda} = 0 \quad \text{and} \quad (f_\alpha \otimes z)^{\langle w^{-1}\alpha^\vee, \lambda \rangle + 1} v_{w\lambda} = 0;$$

- In case $\alpha \in \Delta^+ \cap w\Delta^-$, we have

$$(f_\alpha \otimes z)v_{w\lambda} = 0 \quad \text{and} \quad e_\alpha^{-\langle w^{-1}\alpha^\vee, \lambda \rangle + 1} v_{w\lambda} = 0.$$

In other words, $W(\lambda)_w \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0$ is the generalized Weyl module $W_{w\lambda}$ in the sense of Feigin-Makedonskyi [14].

Proof. The $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ -action is realized by the $U(z\mathfrak{h}[z])$ -action on the highest weight vectors on $W(\lambda)$, and hence so is for each extremal weight vector $v_{w\lambda}$. The other two conditions also hold for $v_{w\lambda} \in W(\lambda)$ by examining possible \mathfrak{h} -weights. As the both modules are cyclic, it follows that we have a \mathfrak{I} -module surjection

$$W_{w\lambda} \twoheadrightarrow W(\lambda)_w \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0. \quad (5.1)$$

Since $W(\lambda)_w$ contains some grading shift of $W(\lambda)$ as its Demazure submodule, we conclude $W(\lambda) \subset W(\lambda)_w \subset W(\lambda)$ as $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ -modules. Here $W(\lambda)$ is a free $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ -module of rank $\dim W(\lambda, 0)$ by Theorem 1.4 2). Therefore, we deduce that $W(\lambda)_w$ is a torsion-free $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ -module of generic rank $\dim W(\lambda, 0)$. By the semicontinuity theorem, we have

$$\dim W_{w\lambda} = \dim W(\lambda, 0) \leq \dim W(\lambda)_w \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0,$$

where the first equality is Feigin-Makedonskyi [14] Theorem B. Therefore, (5.1) forces that above inequality to be an equality. Again by (5.1), we conclude $W_{w\lambda} \cong W(\lambda)_w \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_0$. Moreover, this implies that $W(\lambda)_w$ is $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ -free by (the graded version of) Nakayama's lemma. \square

Let us set $\text{Fl}^{\frac{\infty}{2}}(w)$ to be the quotient of $\mathfrak{Q}(w)$ by the equivalence relation induced from the right $H[[z]]_1$ -action through the embedding $\mathfrak{Q}(w) \subset \mathbf{Q}_G(w)$ (where $H[[z]]_1 := \ker H[[z]] \rightarrow H$).

Corollary 5.2. *We have an isomorphism*

$$\Gamma(\text{Fl}^{\frac{\infty}{2}}(w), \mathcal{O}(\lambda))^* \cong W_{w\lambda}.$$

Proof. The space $\text{Fl}^{\frac{\infty}{2}}(w)$ is a subset of the free quotient of $\mathbf{Q}(w)$ by $H[[z]]_1$ that identifies the two points whose DP-data have finite degrees. Hence, the infinitesimal version of our equivalence relation is realized by $z\mathfrak{h}[z]$. It follows that its global section of a $(G[z]$ -equivariant) line bundle is the $z\mathfrak{h}[z]$ -fixed part of that in $\mathfrak{Q}(w)$. Therefore, Theorem 5.1 implies the result by taking the $z\mathfrak{h}[z]$ -fixed part of $W(\lambda)_w^*$. \square

Remark 5.3. In the previous version of this paper, $\text{Fl}^{\frac{\infty}{2}}(w)$ was the quotient of $\mathbf{Q}_G(w)$ by the right $H[[z]]_1$ -action. In this setting, the proof of Corollary 5.2 was incorrect as [6] is not applicable. In [24], we verified $\Gamma(\mathbf{Q}_G(w), \mathcal{O}(\lambda))^* \cong W(\lambda)_w$. This implies Corollary 5.2 for $\mathbf{Q}_G(w)/H[[z]]_1$ in place of $\text{Fl}^{\frac{\infty}{2}}(w)$. We changed the assertion here in order to keep this paper logically independent from [24].

For each $\gamma \in \Lambda$, we have a polynomial $E_\gamma(q, t) \in \mathbb{C}(q, t)[\Lambda]$ defined in Cherednik [10]. Let us define the bar involution on $\mathbb{C}(q, t)[\Lambda]$ as the ring involution so that $\overline{q^m t^n e^\lambda} := q^m t^n e^{-\lambda}$ for each $m, n \in \mathbb{Z}$ and $\lambda \in \Lambda$. We set $E_\gamma^\dagger(q, t) := \overline{E_\gamma(q, t)}$.

Theorem 5.4 ([14] and [19, 17, 26]). *For $\lambda \in \Lambda_+$, we have*

$$\text{ch } W_{-w_0\lambda} = E_{w_0\lambda}^\dagger(q^{-1}, \infty), \quad \text{and} \quad \text{ch } W_{-\lambda} = E_{w_0\lambda}^\dagger(q, 0).$$

Proof. The first equality is a consequence of Feigin-Makedonskyi [14]. The second equality is proved for type ADE as a combination of Ion [19] and Fourier-Littelmann [17], and in general by Lenart-Naito-Sagaki-Schilling-Shimozono [26] (cf. Chari-Ion [7]). \square

The first equality of the following assertion is [11] Proposition 2.5.

Corollary 5.5. *For $\lambda \in \Lambda_+$, we have equalities*

$$\begin{aligned} D_{w_0}(E_{w_0\lambda}^\dagger(q^{-1}, \infty)) &= E_{w_0\lambda}^\dagger(q, 0) \\ D_{w_0 t_\beta}(E_{w_0\lambda}^\dagger(q, 0)) &= q^{\langle \beta, \lambda \rangle} \cdot E_{w_0\lambda}^\dagger(q^{-1}, \infty), \end{aligned}$$

where $\beta \in Q^\vee$ satisfies $\langle \beta, \alpha_i \rangle < 0$ for each $i \in \mathbf{I}$.

Proof. Taking account into Theorem 5.4, the both assertions follow directly by Theorem 5.1 and Theorem 4.13. \square

6 Non-symmetric Macdonald polynomials

We keep the setting of the previous section. In this section, all cohomologies of (pro-)sheaves are graded \mathfrak{J} -modules obtained from some $\Gamma(\mathcal{Q}(w), \mathcal{O}(\lambda))$ by a finite successive applications of \mathfrak{h} -weight twists and taking cohomologies along \mathbb{P}^1 with making use of vector bundles $M \mapsto SL(2) \times^{B_i^0} M$. Moreover, such operations essentially deal with finitely many distinct \mathfrak{h} -weights when we fix $\lambda \in \Lambda$. Therefore, Theorem 4.12 and the fact that $\text{ch } W(\lambda)_w$ makes sense for each $w \in W$ guarantees the degree-wise Mittag-Leffler condition of the pro-systems defining our sheaves. To this end, we mostly drop the argument on the Mittag-Leffler conditions for the sake of simplicity.

Fix $v \in W$ and a sequence $\mathbf{i} = (i_1, i_2, \dots, i_\ell)$ of elements of \mathbf{I} of length ℓ . We set $w \in W$ to be

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}. \tag{6.1}$$

In case (6.1) is a reduced expression of w , we say that \mathbf{i} is a reduced expression of w . We call that \mathbf{i} is adapted to v if $\ell(wv) = \ell + \ell(v)$ (then \mathbf{i} is a reduced expression of w).

We define

$$\mathcal{Q}(\mathbf{i}, v) := \mathbf{I}_{i_1} \times^{\mathbf{I}} \mathbf{I}_{i_2} \times^{\mathbf{I}} \cdots \times^{\mathbf{I}} \mathbf{I}_{i_\ell} \times^{\mathbf{I}} \mathcal{Q}(v).$$

It induces the multiplication map

$$q_{\mathbf{i}, v} : \mathcal{Q}(\mathbf{i}, v) \ni (g_1, \dots, g_\ell, x) \mapsto g_1 \cdots g_\ell x \in \mathcal{Q}.$$

For each $1 \leq j \leq \ell$, we define a divisor $H_j \subset \mathcal{Q}(\mathbf{i}, v)$ as:

$$H_j = \{(g_1, \dots, g_\ell, x) \in \mathcal{Q}(\mathbf{i}, v) \mid g_j \in \mathbf{I} \subsetneq \mathbf{I}_{i_j}\}.$$

Lemma 6.1. *There exists $u \in W$ so that we have*

$$\mathbb{R}^k(q_{\mathbf{i},v})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},v)} = \begin{cases} \mathcal{O}_{\mathcal{Q}(u)} & (k = 0) \\ \{0\} & (k \geq 1) \end{cases}.$$

Proof. We first prove the case $\ell(w) = 1$. We set $\mathbf{i} = \{i\}$. In case $s_i v < v$, then $\mathcal{Q}(i, v)$ is a \mathbb{P}^1 -fibration over $\mathcal{Q}(i, v)$ through the map $q_{i,v}$ since $\mathbf{I}_i/\mathbf{I} \cong \mathbb{P}^1$. Hence, the assertion holds by setting $u = v$. We consider the case $s_i v > v$. By a similar argument as in Lemma 4.10, we have a map

$$q_{i,s_i v} : \mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v) \longrightarrow \mathcal{Q}(s_i v).$$

The map $q_{i,s_i v}$ is a \mathbb{P}^1 -fibration. The fiber of $q_{i,v}$ along each point of $\mathcal{Q}(s_i v)$ is either pt or \mathbb{P}^1 . By the dimension estimate, we deduce that $\mathbb{R}^k(q_{i,s_i v})_* \mathcal{M} = \{0\}$ ($k \geq 2$) for every \mathbb{G}_m -equivariant pro-coherent sheaf on $\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v)$ satisfying the (degree-wise) Mittag-Leffler condition (or a \mathbb{G}_m -equivariant coherent sheaf on $L_i^0 \times^{B_i^0} \mathcal{Q}(s_i v, \beta)$ for each $\beta \in Q_+^\vee$). We have a short exact sequence

$$0 \rightarrow \ker \rightarrow \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v)} \longrightarrow \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(v)} \rightarrow 0 \quad (6.2)$$

that yields an exact sequence

$$\mathbb{R}^1(q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v)} \longrightarrow \mathbb{R}^1(q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(v)} \rightarrow \mathbb{R}^2(q_{i,s_i v})_* \ker \equiv 0.$$

We have $\mathbb{R}^1(q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v)} = \{0\}$ since $q_{i,s_i v}$ is a \mathbb{P}^1 -fibration. Consequently, we have $\mathbb{R}^1(q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(v)} = \{0\}$. Now the normality of $\mathcal{Q}(s_i v)$ implies $(q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(s_i v)} = (q_{i,s_i v})_* \mathcal{O}_{\mathbf{I}_i \times^{\mathbf{I}} \mathcal{Q}(v)} = \mathcal{O}_{\mathcal{Q}(s_i v)}$, that is the case of $\ell(w) = 1$ (cf. the proof of Theorem 4.12).

We assume the assertion holds for every pair (\mathbf{i}, v) so that the length of \mathbf{i} is $< \ell$ to proceed the induction. We set $\mathbf{i}' = \{i_2, i_3, \dots, i_\ell\}$ and $v' = s_{i_1} u'$, where $u' \in W$ is obtained as u in the assertion for (\mathbf{i}', v) . In case $v' > u'$, we have a factorization

$$\mathcal{Q}(\mathbf{i}, v) \xrightarrow{q^1} \mathcal{Q}(i_1, u') \xrightarrow{q^2} \mathcal{Q}(v') \quad (6.3)$$

so that $q_{i,v} = q^2 \circ q^1$. The induction hypothesis yields $q^1_* \mathcal{O}_{\mathcal{Q}(i_1, u')} = \mathcal{Q}(i_1, u')$ and $\mathbb{R}^k q^1_* \mathcal{O}_{\mathcal{Q}(i_1, u')} = \{0\}$ for $k > 0$. In case $v' < u'$, we have a factorization map obtained from (6.3) by replacing v' with u' . Applying the case $\ell(w) = 1$, the induction (on ℓ) proceeds in the both cases. Therefore, we conclude the assertion by induction. \square

In the below, we denote $u \in W$ determined by the pair (\mathbf{i}, v) by Lemma 6.1 by $u(\mathbf{i}, v)$. For each $j \in [1, \ell]$, we set $\mathbf{i}_j \in \mathbf{I}^{\ell-1}$ be the sequence obtained by omitting the j -th entry, and we set $\mathbf{i}^j \in \mathbf{I}^{\ell-j}$ be the sequence obtained by forgetting the first j entries.

Theorem 6.2 (see [2] Theorem 2.2.6). *Let $v \in W$ be a maximal element so that $v < w$. We have $\ell(v) = \ell(w) - 1$.*

Proposition 6.3. *Let $i \in \mathbf{I}$ and $e \neq w \in W$ so that $s_i w > w$. Let $S_1 \subset W$ be the set of maximal elements so that $v < w$ and $s_i v > v$, and let $S_2 \subset W$ be the set of maximal elements so that $v < w$ and $s_i v < v$. Then, we have*

$$\sum_{v \in S_1 \cup S_2} W(\lambda)_v = \left(\sum_{v \in S_1 \cup S_2} (W(\lambda)_{s_i v} + W(\lambda)_v) \right) \cap W(\lambda)_w \subset W(\lambda)_{s_i w}.$$

Proof. By definition, the \mathfrak{I} -cyclic vector of $W(\lambda)_v$ ($v \in S_1 \cup S_2$) belong to $W(\lambda)_w$, and one of $W(\lambda)_{s_i v}$ ($v \in S_1$), and $W(\lambda)_v$ ($v \in S_2$). Hence the inclusion \subset is clear.

By the proof of Corollary 4.8, we have a uniform basis of $W(\lambda)$ that spans $W(\lambda)_w$ for each $w \in W$. As in the proof of Proposition 4.9, we define $W(\lambda)_{ut_\beta}$ for $u \in W$ and $\beta \in Q_+^\vee$ by twisting the highest weight element of $W(\lambda)$ by q -degree $\langle \beta, w_0 \lambda \rangle$. In view of Naito-Sagaki [31, Theorem 4.2.1], we derive that the vector subspace

$$\left(\sum_{v \in S_1 \cup S_2} (W(\lambda)_{s_i v} + W(\lambda)_v) \right) \cap W(\lambda)_w \subset W(\lambda) \quad (6.4)$$

is spanned by the sum of $W(\lambda)_{ut_\beta}$ for $u \in W$ and $\beta \in Q_+^\vee$ so that ut_β is smaller than the both of $s_i v$ (for some $v \in S_1$) and w with respect to the semi-infinite Bruhat order (see e.g. [27, §6]). Hence, it suffices to prove that an element of W_{aff} covered by the both of w and $s_i v$ with respect to the semi-infinite Bruhat order for some $v \in S_1$ actually belongs to W .

For each $v \in S_1$, we have $v = s_\beta w$ for some $\beta \in \Delta^+$ by Theorem 6.2. By Naito-Sagaki [30, Lemma 2.11] and [27, Lemma 4.1], it suffices to prove that there exists no $\alpha \in \Delta^+$ so that

$$\begin{aligned} \ell(s_\alpha w) - \ell(w) &= 2 \langle \rho, w^{-1} \alpha^\vee \rangle - 1 > 0 \\ \ell(s_i s_\beta s_\alpha w) - \ell(w) &= 2 \langle \rho, w^{-1} \alpha^\vee \rangle - 1 > 0 \\ s_i s_\beta s_\alpha t_{\alpha^\vee} &= s_\gamma t_{\gamma^\vee} \quad \gamma \in \Delta^+. \end{aligned}$$

The last condition forces $s_i s_\beta s_\alpha = s_\alpha$, that is equivalent to $s_i = s_\beta$. Since $v \in S_1$, we have $\beta \neq \alpha_i$. This is a contradiction, and hence an element of W_{aff} covered by the both of w and $s_i v$ belongs to W . Therefore, we conclude that (6.4) is spanned by $W(\lambda)_u$ for $w > u \in W$ as required. \square

Proposition 6.4. *For each $\lambda \in \Lambda$ and $I = [b+1, c] \subset [1, \ell]$ and the pair (\mathbf{i}, v) so that $\ell(u(\mathbf{i}^b, v)) = |I| + \ell(u(\mathbf{i}^c, v))$, we have*

$$H^k(\mathcal{Q}(\mathbf{i}, v), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, v)}(\lambda - \sum_{j=b+1}^c H_j)) = \{0\} \quad k > 0.$$

Proof. We decompose \mathbf{i} into three pieces corresponding to $[1, b]$, $(b, c]$, $(c, \ell]$ as $\mathbf{i}^-, \mathbf{i}^0, \mathbf{i}^+$, respectively.

We set $H_{\mathbf{i}} := \sum_{1 \leq j \leq |\mathbf{i}|} H_j$. We prove the cohomology vanishing assertion by induction on $|\mathbf{i}|$ by assuming $|\mathbf{i}^-| = 0 = |\mathbf{i}^+|$ and $v = e$. The case $|\mathbf{i}| = 0$ is Lemma 6.1 and Theorem 4.12. We have an equality

$$H^0(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_{\mathbf{i}})) = \bigcap_{j \in I} H^0(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j)). \quad (6.5)$$

For each $j \in I$, the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}_j, e)}(\lambda) \rightarrow 0,$$

gives rise to a long exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j)) &\rightarrow H^0(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda)) \\ &\rightarrow H^0(\mathcal{Q}(\mathbf{i}, v), \mathcal{O}_{\mathcal{Q}(\mathbf{i}_j, e)}(\lambda)) \rightarrow H^1(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j)) \rightarrow \dots \end{aligned}$$

Applying Lemma 6.1 and Theorem 4.12, we conclude that

$$\begin{aligned} H^0(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j))^* &\cong W(\lambda)_{u(\mathbf{i}, e)} / W(\lambda)_{u(\mathbf{i}_j, e)}, \quad \text{and} \\ H^{>0}(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_j)) &= \{0\} \end{aligned}$$

for each $j \in I$. Note that the assertion follows from this when \mathbf{g} is of type A_1 .

The induction hypothesis yields the desired cohomology vanishing if we replace \mathbf{i} with \mathbf{i}' obtained from \mathbf{i}' by omitting the first entry i_1 . In view of [2, Corollary 2.2.3], each element of the set S of maximal elements in W that is smaller than w is realized as $u(\mathbf{i}'_j, v)$ for some $1 \leq j \leq \ell - 1$. Since we have $W(\lambda)_x \subset W(\lambda)_y$ if $x < y \in W$, we can omit j from the RHS of (6.5) so that $u(\mathbf{i}'_j, v) \notin S$. By Proposition 6.3, we deduce that

$$\sum_{x \in S} W(\lambda)_x = \left(\sum_{x \in S} W(\lambda)_x + W(\lambda)_{s_{i_1}x} \right) \cap W(\lambda)_{u(\mathbf{i}', e)} \subset W(\lambda)_{u(\mathbf{i}', e)} \subset W(\lambda)_{u(\mathbf{i}, e)} \quad (6.6)$$

is the intersection of a graded $HL_{i_1}^0$ -submodule with $W(\lambda)_{u(\mathbf{i}', v)}$. In view of Lemma 4.4 (cf. the proof of Corollary 4.8 and Theorem 4.13), the associated graded of (6.6) gives the direct sum of embeddings of $B_{i_1}^0$ -modules of the form:

$$\{0\} \subset \{0\}, \{0\} \subset \mathbb{C}_{m\varpi}, \mathbb{C}_{m\varpi} \subset \mathbb{C}_{m\varpi}, \quad \text{and} \quad V(m\varpi) \subset V(m\varpi) \quad \text{inside} \quad V(m\varpi), \quad (6.7)$$

where $V(m\varpi)$ is the $(m+1)$ -dimensional irreducible module of $L_{i_1}^0$, and $\mathbb{C}_{m\varpi}$ is its $B_{i_1}^0$ -eigenspace. Since we have

$$H^k(\mathcal{Q}(\mathbf{i}', e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}', e)}(\lambda - \sum H_{i'}))^* \cong \begin{cases} W(\lambda)_{u(\mathbf{i}', e)} / \sum_{x \in S} W(\lambda)_x & (k = 0) \\ \{0\} & (k > 0) \end{cases},$$

we deduce from (6.7) with \mathbb{P}^1 -calculations to derive

$$H^1(\mathcal{Q}(\mathbf{i}, e), \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_i)) \cong H^1(\mathbb{P}^1, \mathcal{F}(-1)) = \{0\},$$

where \mathcal{F} is the $HL_{i_1}^0$ -equivariant vector bundle induced from the graded $HB_{i_1}^0$ -module $W(\lambda)_{u(\mathbf{i}', e)} / \sum_{x \in S} W(\lambda)_x$. Therefore, the long exact sequence associated to the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_i) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}, e)}(\lambda - H_{i'}) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{i}', e)}(\lambda - H_{i'}) \rightarrow 0$$

(where $H_{i'}$ on $\mathcal{Q}(\mathbf{i}, e)$ is the inflation of $H_{i'}$ on $\mathcal{Q}(\mathbf{i}', e)$), together with the induction hypothesis implies the desired cohomology vanishing. This proceeds the induction, and consequently we have obtained the assertion when $|\mathbf{i}^-| = 0 = |\mathbf{i}^+|$ and $v = e$.

In view of the above discussion using (6.5), Lemma 4.4, and Corollary 4.8, adding one element in the beginning of \mathbf{i}^- amounts to form the corresponding vector bundles and then take its cohomology. In this case, the associated graded of an analogous filtration on

$$\sum_{b < j \leq c} W(\lambda)_{u(\mathbf{i}'_j, e)} \subset W(\lambda)_{u(\mathbf{i}', e)} \subset W(\lambda)_{u(\mathbf{i}, e)}$$

adds some more cases $(\mathbb{C}_{m\varpi} \subsetneq V \subsetneq V(m\varpi) \subset V(m\varpi))$ to (6.7) with vanishing higher cohomologies. This is harmless (as we have no degree -1 twist) in this case, and we conclude the result by induction when $|\mathbf{i}^+| = 0$ and $v = e$.

Now we consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{j}',e)}(\lambda - \sum_{j=b+1}^{\ell-m} H_j) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{j}',e)}(\lambda - \sum_{j=b+1}^{\ell-m-1} H_j) \rightarrow \mathcal{O}_{\mathcal{Q}(\mathbf{j}'',e)}(\lambda - \sum_{j=b+1}^{\ell-m-1} H_j) \rightarrow 0,$$

where \mathbf{j}' is the sequence formed by the first $(\ell-m)$ letters in \mathbf{i} , and \mathbf{j}'' is formed by the first $(\ell-m-1)$ letters in \mathbf{i} . By arguing from the case $m=0$ (corresponding to $|\mathbf{i}^+|=0$), a repeated use of the long exact sequences implies the result for $v=e$ by induction.

We can factor $q_{\mathbf{i},e}$ into the composition of $q_{\mathbf{i}^-, \mathbf{i}^0, u(\mathbf{i}^+, e)}$ and the inflation of $q_{\mathbf{i}^+, e}$. Hence, the Leray spectral sequence gives the result from Lemma 6.1 as required. \square

Corollary 6.5. *For each $\lambda \in \Lambda$ and $I = (b, c] \subset [1, \ell]$ so that $\ell(u(\mathbf{i}^b, v)) = |I| + \ell(u(\mathbf{i}^c, v))$, we have*

$$\mathbb{R}^k(q_{\mathbf{i},v})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},v)}(-\sum_{j \in I} H_j) = \{0\} \quad \text{for each } k > 0.$$

Proof. In view of Lemma 2.7, the assertion follows from the projection formula and Proposition 6.4 by the Leray spectral sequence. \square

In the below, we assume that \mathbf{i} is a reduced expression of w unless stated otherwise. Note that the assumption of Corollary 6.5 holds automatically. For each $\lambda \in \Lambda_+$, we set

$$\mathcal{E}_w(\lambda) := (q_{\mathbf{i},e})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},e)}(\lambda - \sum_{k=1}^{\ell} H_k) \cong \left((q_{\mathbf{i},e})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},e)}(-\sum_{k=1}^{\ell} H_k) \right) \otimes_{\mathcal{O}_{\mathcal{Q}}} \mathcal{O}_{\mathcal{Q}}(\lambda).$$

We have a natural inclusion $\mathcal{E}_w(\lambda) \subset \mathcal{O}_{\mathcal{Q}(w)}(\lambda)$ defined as:

$$\mathcal{E}_w(\lambda) \equiv (q_{\mathbf{i},e})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},e)}(\lambda - \sum_{k=1}^{\ell} H_k) \hookrightarrow (q_{\mathbf{i},e})_* \mathcal{O}_{\mathcal{Q}(\mathbf{i},e)}(\lambda) \equiv \mathcal{O}_{\mathcal{Q}(w)}(\lambda). \quad (6.8)$$

By construction, each $\mathcal{E}_w(\lambda)$ is $(\mathbf{I} \times \mathbb{G}_m)$ -equivariant.

Lemma 6.6. *For each $w \in W$ and $\lambda \in \Lambda_+$, the sheaf $\mathcal{E}_w(\lambda)$ is independent of the choice of a reduced expression of w .*

Proof. It is enough to check that two reduced expressions of w gives rise to the same sheaf. As the construction is about the modification along the fiber, the assertion reduces to the following assertion (as $q_{\mathbf{i},e}$ factors through a $X(\mathbf{i})$ -fibration): We define

$$X(\mathbf{i}) := P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_{\ell}} / B$$

and consider the natural map $q_{\mathbf{i}} : X(\mathbf{i}) \rightarrow X$. We define H_k ($1 \leq k \leq \ell$) to be the divisor defined by the pullback of H_k through $X(\mathbf{i}) \hookrightarrow \mathcal{Q}(\mathbf{i}, e)$, and set $\Sigma_{\mathbf{i}} = \sum_{k=1}^{\ell} H_k$. Then, the direct image $(q_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}(-\Sigma_{\mathbf{i}})$ does not depend on \mathbf{i} whenever \mathbf{i} is a reduced expression of the same $w \in W$.

Thanks to the braid relation, this further reduces to the case $w = w_0$ and G is rank two. In addition, $(q_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}(-\Sigma_{\mathbf{i}})$ is the kernel of the map $\mathcal{O}_{X(\mathbf{i})} \rightarrow$

$\bigoplus_{\mathbf{j}} \mathcal{O}_{X(\mathbf{j})}$, where \mathbf{j} runs over all the subwords. Therefore, taking the dual of the global sections yields

$$\Gamma(X, (q_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}(\lambda - \Sigma_{\mathbf{i}}))^* \cong \text{coker} \left(\bigoplus_{v < w_0} V(\lambda)_v \rightarrow V(\lambda) \right)$$

for each $\lambda \in \Lambda_+$. Thanks to Theorem 4.2 4), each of the map $V(\lambda)_v \rightarrow V(\lambda)$ is the natural inclusion. Hence, $\Gamma(X, (q_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}(\lambda - \Sigma_{\mathbf{i}}))^*$ is independent of the choice of \mathbf{i} . For $\lambda \gg 0$, the sheaf $(q_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}(\lambda - \Sigma_{\mathbf{i}})$ is generated by its global sections. Therefore, we conclude the result. \square

Lemma 6.7. *For each $\lambda \in \Lambda_+$ and $w \in W^\lambda$, the module $\Gamma(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^*$ has a \mathfrak{J} -cyclic vector with its \mathfrak{h} -weight $w\lambda$.*

Proof. By construction, we have an inclusion $\mathcal{E}_w(\lambda) \subset \mathcal{O}_{\mathcal{Q}(w)}(\lambda)$. This results a surjection $W(\lambda)_w \rightarrow \Gamma(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^*$ of \mathfrak{J} -modules by taking the dual of their global sections. Since $W(\lambda)_w$ is a \mathfrak{J} -module with a cyclic vector of weight $w\lambda$, so is $\Gamma(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^*$. \square

For $\lambda \in \Lambda_+$, we set $W_\lambda := \langle s_i \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \rangle$. We define W^λ to be the set of minimal length representatives of the coset W/W_λ .

Theorem 6.8 (Cherednik-Orr [11] Proposition 2.5). *Let $\lambda \in \Lambda_+$ and let $w \in W^\lambda$ so that $s_i w > w$ and $s_i w \in W^\lambda$ for some $i \in \mathbf{I}$. Then, we have:*

1. *If $w^{-1}\alpha_i = \alpha_j$ for some $j \in \mathbf{I}$ so that $\langle \alpha_j^\vee, \lambda \rangle > 0$, then we have*

$$(1 - q^{\langle \alpha_j^\vee, \lambda \rangle}) E_{-s_i w \lambda}^\dagger(q^{-1}, \infty) = D_i \left(E_{-w \lambda}^\dagger(q^{-1}, \infty) \right) - E_{-w \lambda}^\dagger(q^{-1}, \infty);$$

2. *If $w^{-1}\alpha_i$ is not a simple root, then we have*

$$E_{-s_i w \lambda}^\dagger(q^{-1}, \infty) = D_i \left(E_{-w \lambda}^\dagger(q^{-1}, \infty) \right) - E_{-w \lambda}^\dagger(q^{-1}, \infty).$$

Proof. If we set $T_i := D_i - 1$, then the adjoint of the bar-involution yields the Hecke operator T_i specialized to $t = \infty$ (see e.g. [29] 1st ver. 5.1). Therefore, the current formulation is equivalent to [11] Proposition 2.5. \square

Proposition 6.9. *For each $\lambda \in \Lambda_+$ and $w \in W^\lambda$, we define $\lambda_w := \lambda - \sum_{w\alpha_j < 0} \varpi_j$. Then, we have*

$$\sum_{i \geq 0} (-1)^i \text{ch } H^i(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^* = \left(\prod_{i \in \mathbf{I}} \prod_{k=1}^{\langle \alpha_i^\vee, \lambda_w \rangle} \frac{1}{1 - q^k} \right) \cdot E_{-w \lambda}^\dagger(q^{-1}, \infty).$$

Proof. We define the (dual) Euler-Poincaré characteristic of an $(\mathbf{I} \times \mathbb{G}_m)$ -equivariant (pro-)coherent sheaf \mathcal{F} by

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \text{ch } H^i(\mathcal{Q}, \mathcal{F})^* \in \mathbb{Q}((q))[\Lambda] \cup \{\infty\},$$

where we understand it to be ∞ if one the coefficient of a monomial is ∞ .

We prove the assertion on induction on w . The case $w = e$ is Theorem 5.4. Hence, we assume the assertion for every $v < w$ to deduce the assertion for w . For $i \in \mathbf{I}$ so that $s_i w > w$, we set $H := H_1$ and $q_i := q_{i,w}$ for simplicity. By Corollary 6.5, we have a short exact sequence

$$0 \rightarrow (q_i)_* \mathcal{E}_w^+(-H) \rightarrow (q_i)_* \mathcal{E}_w^+ \rightarrow \mathcal{E}_w \rightarrow 0,$$

where we denote \mathcal{E}_w^+ the inflation of \mathcal{E}_w from $\mathcal{Q}(w)$ to $\mathcal{Q}(i, w)$. Now we have

$$\chi((q_i)_* \mathcal{E}_w^+(-H)) = D_i(\chi(\mathcal{E}_w)) - \chi(\mathcal{E}_w). \quad (6.9)$$

In case $w^{-1}\alpha_i = \alpha_j$ for some $j \in \mathbf{I}$, then we have $\lambda_{s_i w} = \lambda_w - \varpi_j$. Therefore, the comparison of (6.9) and Theorem 6.8 1) proceeds the induction.

In case $w^{-1}\alpha_i \notin \Pi$, then we have $\lambda_{s_i w} = \lambda_w$. Therefore, the comparison of (6.9) and Theorem 6.8 2) proceeds the induction.

These proceed the induction in the both cases as required. \square

Corollary 6.10. *For each $\lambda \in \Lambda_+$ and $w \in W^\lambda$, we define $\lambda_w := \lambda - \sum_{w\alpha_j < 0} \varpi_j$. Then, we have*

$$\mathrm{ch} H^i(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^* = \begin{cases} \left(\prod_{i \in \mathbf{I}} \prod_{k=1}^{\langle \alpha_i^\vee, \lambda_w \rangle} \frac{1}{1-q^k} \right) \cdot E_{-w\lambda}^\dagger(q^{-1}, \infty) & (i = 0) \\ 0 & (i > 0) \end{cases}.$$

Proof. By setting \mathbf{i} to be adapted to e , Proposition 6.4 and Corollary 6.5 implies

$$H^i(\mathcal{Q}(w), \mathcal{E}_w(\lambda))^* = \{0\} \quad i > 0$$

Therefore, Proposition 6.9 yields the result. \square

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